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WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

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1920. (The Editor.)—If in the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0,$$

we have e^r, or e^{hr}, or $e^{\frac{1}{h}r}$ as a root; show that, in the three cases respectively, we shall have the relations

$$\left.\begin{aligned} p_0 + p_2 + p_4 + \&c. &= p_1 + p_3 + p_5 + \&c. \\ p_1 + p_5 + p_9 + \&c. &= p_3 + p_7 + p_{11} + \&c. \\ p_0 + p_4 + p_8 + \&c. &= p_2 + p_6 + p_{10} + \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(2),$$

$$\left.\begin{aligned} p_1 + p_3 + p_7 + p_8 + \&c. &= p_4 + p_6 + p_{10} + p_{11} + \&c. \\ p_1 - p_2 - p_4 + p_5 + \&c. &= 2(p_0 - p_3 + p_6 - p_9 + \&c.) \end{aligned}\right\} \ldots\ldots(3). \quad 86$$

1987. (Dr. Ball.)—If α, β, γ, δ be the roots of $(a, b, c, d, e)(x, 1)^4 = 0$, and α', β', γ', δ' the roots of $(a_1, b_1, c_1, d_1, e_1)(x, 1)^4 = 0$, find equation which has for its roots all the different values of the

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 31$$

2011. (The Editor.)—Show that the trilinear equation of the circle which passes through the points where the internal bisectors of the angles of the triangle of reference meet the sides, is

$$(u + \cos A)\alpha^2 + (v + \cos B)\beta^2 + (w + \cos C)\gamma^2 = (v + w + \cos B + \cos C)\beta\gamma + (u + w + \cos A + \cos C)\gamma\alpha + (u + v + \cos A + \cos B)\alpha\beta,$$

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2105. (W. S. B. Woolhouse, F.R.A.S.)—Prove that

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2108. (S. Bills.)—One sovereign, two half-sovereigns, one crown, two half-crowns, one shilling, two sixpences, one penny, and two half-pennies, are put into a bag, from which three persons, A, B, C, draw out, in succession, each of them a coin. Show that the value of the expectation of each is $4s.\ 4\frac{1}{3}d.$ 44

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2215. (The Editor.)—Produce a given straight line, so that the square on the whole line thus produced may be to the rectangle contained by the given line and the part produced in a ratio which is given, or a minimum. 101

2216. (The Editor.)—From any point on an ellipse or hyperbola draw two tangents to the circle on the minor axis as diameter; then, if from the ends of the diameter conjugate to that passing through the given point parallels be drawn to the tangents, prove that they will pass through the foci. 91

2218. (The Editor.)—Find the point at which a triangle must be fixed; so that when it is struck at an angle by a blow perpendicular to its plane, it may begin to revolve about an axis parallel to the opposite side. 94

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3143. (T. P. Kirkman, M.A., F.R.S.)—

Soon after Easter,
As every feaster
In Parliament can tell,
To every and to all his well-beloved Peers and Commons
Mr. Punch presents
His compliments
And customary summons.

He admits to his levée and breakfast-board
Along with the Ministers and P.C.'s,
Who go, of course, whenever they please,
The other Senators in threes,
Three daily for just eight weeks;
And you may imagine how every Lord
And rising Commoner keenly seeks,
By hook or crook, to beg or buy admission in the morning triad.

"He has been thrice,
And I but twice?"—
Great is the din;
Teazing the Chancellor,—
Mobbing the Speaker,—
What name to cancel, or
Which to put in;
Who for next week, or
Who's down for to-morrow;
Whom to leave out
For a year to pout
In envious sorrow.

The Chancellor trims the triplets so,
While Peers implore,
And Commons roar,
That never a pair meet twice at the feast;
That all who go,
Go thrice at least,
And no one gets more
Than breakfasts four.

You have seen Mr. Punch
At dinner and lunch:
This yarn of his breakfast,
Will soon have your neck fast,
Unless you can check fast.

The fifty-six threes
Are formed without ease.
Can you do it for me,
And supply the key
Mathematically?
You've data enough,
If *quantum suff.*
Of other stuff.

..... 71

3229. (Professor Townsend, F.R.S.)—A material particle, moving freely under the action of two attractive or repulsive forces emanating from fixed centres, and varying inversely as the square of the distance, sets out without initial velocity from any point on the surface of the sphere coaxal with the two centres of force whose centre divides the interval between them externally in the ratio of the absolute forces; determine the orbit it describes. 17

3279. (J. J. Sylvester, F.R.S.)—If a quadratic equation with real coefficients be written down at hazard, show that the probability of its roots being imaginary is $\frac{3}{2}\frac{1}{2} - \frac{1}{12}\log 2$, or ·3727932. 25

3400. (Artemas Martin.)—Show that the average area of all the hexagons that can be formed by cutting off the corners of a given triangle, the points of section being uniformly distributed over the sides, is two-thirds of the area of the triangle. 64

3467. (S. Watson.)—Two points are taken at random within a given rectangle; show that the chance of the straight line passing through them intersecting opposite sides is $\frac{2}{3}$. 106

3514. (The Editor.)—Two points are taken at random in each of the opposite sides of a quadrilateral; find the probability that the quadrilateral of which they are the vertices will be less than half the given quadrilateral..... 26

3648. (The Editor.)—Show (1) that the pedal of a four-cusped hypocycloid is a curve which may be defined by the corresponding equations in polar and rectangular coordinates

$$r = a\sin 2(\theta + \alpha) - c\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$\text{or } (x^2+y^2+cx)^2(x^2+y^2) = a^2\left\{(x^2-y^2)\sin 2\alpha + 2xy\cos 2\alpha\right\} \ldots(\alpha');$$

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$2a$ being the radius of the fixed circle from which the hypocycloid is generated, c the distance of the pedal origin O from the centre Q, and α the inclination of QO to a cuspidal tangent; (2) that when the origin O is at the centre Q, the pedal is the four-branched *corolla*

$$r = a \sin 2\theta \ldots\ldots (\beta), \text{ or } (x^2+y^2)^3 = 4a^2x^2y^2 \ldots\ldots (\beta');$$

(3) that when the origin O is in a radius QA drawn to a vertex A of the hypocycloid, the pedal is the four-branched *scarabæus*

$$r = a\cos 2\theta - c\cos\theta \ldots\ldots (\gamma),$$

$$\text{or } (x^2+y^2+cx)^2(x^2+y^2) = a^2(x^2-y^2)^2 \ldots\ldots (\gamma');$$

and show further (4, 5) that when O is between Q and A, this pedal has four real branches passing through O and forming two unequal loops that touch two branches of the hypocycloid at opposite vertices (A, B), and two equal loops that touch the other two branches of the hypocycloid at points (E, F) on the same side of the vertices with the origin; but (6) if O is at the vertex A, two of the branches of the pedal become there a cusp which, near the origin, takes the form of the semi-cubical parabola $2x^3 + 3ay^2 = 0$; and (7) when O is outside the hypocycloid, the origin is a conjugate point through which only two real branches pass. Also show (8) that the area of any loop of the pedal may be obtained from the formula

$$\tfrac{1}{4}a^2\left\{(1+e^2)\theta - 3e\sin\theta(7-\cos^2\theta)\right\},$$

and that the entire area of the pedal is $\frac{1}{4}(1+e^2)\pi a^2$. 59

3662. (G. S. Carr.)—A rectangular lamina of length a, width b, and thickness c, elastic only to flexure in a plane perpendicular to its width, is to be cut so that when A, B, the middle points of its ends, are attached to a string of length $d < a$, it may form the cylindrical surface whose equation, with A for origin and AB for axis of x, is $y = f(x)$. Show generally how to determine its shape; first, when the width only is varied, and b is its maximum value; secondly, when the thickness only is altered, and c, which is but small, is allowed for its maximum value. 32

3696. (G. S. Carr.)—A uniform rod of length a, elastic only to flexure, has its ends fastened to a string of length $b < a$; determine the nature of the curve. 41

3727. (G. S. Carr.)—Show that the shape of a uniform plane elastic lamina, which will take a circular form when its ends are drawn together to meet, may be obtained by drawing two cutting planes through a tangent which is at right angles to the axis of a cylinder, and unrolling the intercepted part of the surface. 44

3731. (S. Watson.)—EG is a focal chord in an ellipse, and MP a perpendicular to it at its middle point M. Show that the curve which the line MP always touches is a three-cusped curvilinear triangle, to which the axes of the ellipse are tangents, and that its area is to that of the ellipse as $e^6 : 1-e^2$, where e is the excentricity. 28

3755. (S. Watson.)—From any point P in a diameter of a given sphere, perpendiculars PQ, PR are drawn of any length and

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	in any direction, so that P and Q may lie within the sphere; and through P, Q, R two spheres are drawn to touch the given one; show that the sum of their radii is equal to the radius of the given sphere.....	39
3806.	(Professor Townsend, F.R.S.)—Defining, as the harmonic cone of intersection of two quadrics with respect to a point, the cone of the second order generated by the system of lines through the point, which are cut harmonically by the quadrics; show that the harmonic cone of intersection of any two quadrics U and V, with respect to any point, α', β', γ', δ', passes through the four lines of intersection of the two cones which, from the point as vertex, envelope any two quadrics $U+kV$ and $U-kV$ concurrent with and harmonically conjugate to each other with respect to U and V.	93
3810.	(G. S. Carr.)—Show that the law of central force for a particle describing the orbit $x^n+y^n=1$ (n being an even integer) is $F \propto x^{n-2}y^{n-2}(x^2+y^2)^{\frac{1}{2}}$. When $n=\infty$ the orbit becomes a square, and the velocity at the corners is to that elsewhere as $1:\sqrt{2}$.....	82
3815.	(J. J. Walker, M.A.)—Show that the locus of feet of perpendiculars let fall from points in a given diameter of a quadric surface on the polar planes of those points is a rectangular hyperbola.	45
3816.	(N'Importe.)—Given the point of contact of one of two fixed tangents to a parabola; show that the locus of the focus is a circle passing through the intersection of the tangents, and touching one of them.	75
3834.	(J. J. Walker, M.A.)—Show (1) that any two isochronous straight lines of descent from a given point above an inclined plane to the plane are equally inclined to that of quickest descent (all being drawn in the same vertical plane); and (2) that the velocity acquired in sliding down the last is a mean proportional between the velocities acquired in sliding down the first-named lines.....	66
3843.	(M. Collins, B.A.)—If a whole number m divides a^n-1; prove that m^2 divides $a^{mn}-1$, that m^3 divides $a^{m^2n}-1$, and that m^{r+1} divides $a^{m^rn}-1$.....	50
3850.	(The Editor.)—Prove that the side of a square inscribed in one of the segments of a circle cut off by the side of an inscribed equilateral triangle of side a is $\frac{1}{78}(\sqrt{57}-2\sqrt{3})\,a$, or $(\cdot27238)\,a$.	42
3852.	(Professor Townsend, F.R.S.)—Show that a variable tangent to the curve of intersection of any two fixed quadrics determines with the four vertices of their common self-reciprocal tetrahedron a pencil of four planes having a constant anharmonic ratio.....	50
3876.	(J. J. Sylvester, F.R.S.)—Show that there are 3184 positions in a cubic curve such that at each of them curves of the 50th order may be drawn having 90-point contact with the cubic....	46

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3882. (Professor Townsend, F.R.S.)—The circumscribed and inscribed circles of a variable triangle, plain or spherical, being supposed both fixed; show that, throughout the deformation of the triangle, velocity of A : velocity of B : velocity of C = $\cot\frac{1}{2}A : \cot\frac{1}{2}B : \cot\frac{1}{2}C$ in either case; and hence that, angular velocity of a : angular velocity of b : angular velocity of c = $a : b : c$ for the plane, and = $\tan\frac{1}{2}a : \tan\frac{1}{2}b : \tan\frac{1}{2}c$ for the spherical triangle. 29

3884. (W. Godward.)—A point C is taken in the diameter AB of a semicircle, and on AC, BC other semicircles are described; also a series of circles whose centres are P_1, P_2, P_3 ..., are constructed such that each one has contact with the preceding and the semicircles on AB, AC; the first circle P_1 having contact with the semicircle on BC. Prove that the loci of P_1, P_2, P_3 ... are ellipses having a common focus. 88

3888. (J. Collins.)—AA_1, BB_1, CC_1 are perpendiculars from the vertices A, B, C of a given triangle upon a straight line given in position, and A_1A_2, B_1B_2, C_1C_2 are perpendiculars upon the sides opposite the vertices A, B, C. Prove that the last three perpendiculars pass through the same point. 80

3889. (Rev. Dr. Booth, F.R.S.)—A right angle moves along a conchoid, one side passes through the pole; find the curve enveloped by the other side. 21

3892. (J. J. Walker, M.A.)—Show that the locus of the intersection of tangents at the extremities of a normal chord of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$. 41

3894. (C. Taylor, M.A.)—Prove that the polar of a point with respect to a conic meets the directrices on the tangents from that point to a confocal. 38

3901. (S. Bills.)—Given the base, the vertical angle, and the distance from the centre of the circumscribed circle to the centroid of a plane triangle; to construct the triangle. 43

3909. (Professor Townsend, F.R.S.)—The interior of a homogeneous shell, bounded internally and externally by similar and similarly placed but not concentric ellipsoids, and attracting according to the law of the inverse square of distance, is partially filled with homogeneous fluid; given all particulars, determine the form and position of the free surface of the latter under the action of the former. 105

3910. (F. D. Thomson, M.A.)—Show (1) that the three diagonals of any rectilinear hexagon drawn on the surface of an hyperboloid meet in a point; and (2) deduce therefrom Brianchon's Theorem for a plane conic. 65

3918. (Rev. W. Roberts, M.A.)—Let a polygon be inscribed in an equilateral hyperbola: prove that the algebraic sum of each side, multiplied by the distance of its middle point from the centre, is equal to zero. 30

3921. (P. O. Cavanagh.)—Find the six intersections real or imaginary of Descartes' folium $x^3-3axy+y^3=0$ with the ellipse $x^2-xy+y^2-a(x+y)+a^2=0$. 50

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3922. (A. Martin.)—Show (1) that the average area of all the triangles whose bases = a, and sum of the other two sides = b, is $\frac{1}{16}\pi a\,(b^2-a^2)^{\frac{1}{2}}$, and (2) that the average area of the circles inscribed in them is $\frac{\pi a^2}{6}\left(\frac{b-a}{b+a}\right)$. 70

3923. (The Editor.)—(1) From the corners of a triangle cut three triangles of given areas, so that the remainder may be a triangle; also solve the analogous problems for (2) a square, (3) a polygon, and (4) a tetrahedron; and give a numerical example in each case..... 22

3925. (G. O'Hanlon.)—P is any point in the base BC of the right-angled triangle ABC, and BD, PF are drawn perpendicular to the hypotenuse AC: prove that $AF.AD+BD.PF=AB^2$. ... 42

3928. (J. J. Sylvester, F.R.S.)—If a particle solicited by two attractive forces, varying as the inverse squares of the distances from two fixed points, and by a force varying as the direct distance from a third fixed point midway between the two others, start from rest in the direction of a tangent to the hyperbola, of which the two first named points are the foci, prove that it will continue to oscillate in an arc of the hyperbola..... 17

3931. (J. M. Wilson, M.A.)—Show that, in general, eight spheres (and eight only) can be drawn to touch four planes which intersect so as to form a tetrahedron..... 20

3932. (The Editor.)—From a bag containing n marbles a number is taken at random and put into another bag; show (1) that if a number be taken at random from the second bag, the respective probabilities of obtaining an odd and an even number are as

$$\Sigma_{r=1}^{r=n}\frac{C(n,r)}{2^r-1}+2^n-1 : 2^n-1-\Sigma_{r=1}^{r=n}\frac{C(n,r)}{2^r-1},$$

where $C(n,r)$ denotes the number of combinations of n things taken r at a time; and hence show (2) that if $n=12$, these probabilities are as

86872555317477712 : 80838947728038263..... 29

3936. (S. Tebay, B.A.)—Along the arc of a circle n equal particles are distributed at equal distances, the first being fixed. Find the locus of their centre of gravity, when the distance between the particles is supposed to vary..... 108

3937. (T. T. Wilkinson, F.R.A.S.)—Two similar polygons, whose areas are (A) and (B), are so situated that (B) is enclosed by (A), and their corresponding sides are parallel. If a third polygon (C) be inscribed to (A) and circumscribed to (B), prove that $(C)^2=(A).(B)$..... 26

3945. (F. D. Thomson, M.A.)—P, P′ are fixed points on one of a series of confocal conics, and from P, P′ tangents are drawn to another conic of the series, meeting the first conic in Q, R; Q′, R′. Show that the locus of the intersection of QR, Q′R′ is a conic..... 51

3946. (Matthew Collins, B.A.)—Find (1) the five-point parabolic asymptote of the quartic curve

$$x^2y^2+axy^2+bx^2y+cx^3+d^2xy+e^2x^2+f^3y=0,$$

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(2) the coordinates of the centre of gravity of the triangle formed by the other three points common to both curves, and (3) the equation of its circumscribed circle. 40

3947. (S. Watson.)—If O_1, O_2, O_3 be the centres of the escribed circles of the triangle ABC, and ρ_1, ρ_2, ρ_3 the radii of circles inscribed in the triangles BO_1C, CO_2A, AO_3B; prove that

$$\left(\frac{a}{\rho_1}+\frac{b}{\rho_2}+\frac{c}{\rho_3}\right)^{\frac{1}{2}}=\left(\frac{a}{r_1}\right)^{\frac{1}{2}}+\left(\frac{b}{r_2}\right)^{\frac{1}{2}}+\left(\frac{c}{r_3}\right)^{\frac{1}{2}}. \quad \text{.....} \quad 43$$

3956. (The Rev. Dr. Booth, F.R.S.)—Two planes are given in position, and a point equidistant from them. Find the locus of a point whose distance from the former shall be equal to the square root of the product of the perpendiculars on the given planes from the moving point. 62

3957. (Professor Townsend, F.R.S.)—Show that a material particle, moving freely in the plane of a thin uniform circular ring, whose elements attract it according to the law of the inverse cube of the distance, will, if projected with the velocity just sufficient to carry it to infinity against the action of the resultant force, describe the arc of the epicycloid (or hypocycloid) concentric with the ring, which commences at the point in the direction of projection, and terminates at the first cusp on the circumference of the circle. 36

3959. (J. J. Sylvester, F.R.S.) — Prove (1) that the equation $x^4+4ex^3+6e^2x^2+4ex+1=0$ has no real roots if $\pm e$ lies without the limits $\frac{1}{3}$ and 1, and has two real and two imaginary roots if $\pm e$ lies within these limits. Prove also (2) that the equation $x^5+5ex^4+10e^2x^3+10e^2x^2+5ex+1=0$ has only one real root (viz., $x=-1$) if e lies without the limits $\frac{1}{3}$ and 1, and has two real and two imaginary roots if e lies within these limits. 87

3960. (The Editor.)—From a point P in the plane of a triangle ABC perpendiculars PD, PE, PF are drawn on the sides; prove (1) that the average area of the triangle DEF is one-eighth of the triangle ABC if P be allowed to range over the circle drawn round ABC, but (2) if P be limited to points within the triangle ABC, the average is

$$\tfrac{1}{12}(\sin^2 A+\sin^2 B+\sin^2 C)\,\Delta, \quad \text{or} \quad \tfrac{1}{8}(1+\cos A\cos B\cos C)\,\Delta,$$

$$\text{or} \quad \frac{1}{12}\left(\frac{a^2+b^2+c^2}{4R^2}\right)\Delta,$$

where Δ is the area of the triangle, and R the radius of the circumscribed circle. 33

3962. (J. J. Walker, M.A.)—If A, B, C, D be four points whose distances asunder are $BC=a$, $CA=b$, $AB=c$, $DA=a'$, $DB=b'$, $DC=c'$; prove that, whether these points are in the same plane or are the corners of a tetrahedron, we shall have the relations

$$2aa'\cos(aa')=(b^2+b'^2)-(c^2+c'^2),\ 2bb'\cos(bb')=(c^2+c'^2)-(a^2+a'^2),$$

$$2cc'\cos(cc')=(a^2-a'^2)-(b^2+b'^2) \quad \text{..... (1)},$$

$$aa'\cos(aa')+bb'\cos(bb')+cc'\cos(cc')=0 \quad \text{..... (2)},$$

and (3) if the four points are in the same plane, sixteen times the square of the area of the quadrilateral ABCD is equal to

$$(a^2+a'^2+2bb'-c^2-c'^2)(c^2+c'^2+2bb'-a^2-a'^2). \quad \text{.....} \quad 56$$

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3966. (Dr. Casey.)—Prove that (1) if any two circles be inverted from an arbitrary point, the ratio of the square of their common tangent to the rectangle of their diameters is the same for the converse as for the original circles; (2) employ the preceding theorem in proving Terquem's theorem on the nine-point circle, giving a proof that will apply equally whether the triangle be formed by right lines or by arcs of circles; and (3), prove Terquem's theorem for spherical triangles formed by arcs of great or small circles..... 54

3971. (C. Taylor, M.A.)—The diagonals of a quadrilateral intersect at right angles in O. Show how to inscribe a conic with O as focus. 69

3974. (Rev. R. Harley, F.R.S.)—Required (1) an explicit solution of the equation

$$(1+ax^2)\frac{d^2y}{dx^2} + ax\frac{dy}{dx} - n^2y = 0;$$

and (2) deduce therefrom the solution of the equation

$$x^2(x^2+a)\frac{d^2y}{dx^2} + x(2x^2+a)\frac{dy}{dx} - n^2y = 0. \quad \ldots\ldots\ldots 33$$

3978. (Rev. Dr. Booth, F.R.S.)—Prove that the volume of any slice of a surface of the second order, bounded by parallel planes, may be found by the formula $V = \frac{1}{2}t(A+B) + \left(\frac{abc}{p^3}\right)\frac{1}{6}(t^3\pi)$, where A and B are the areas of the parallel slices, t the thickness of the slice, and p the perpendicular from the centre on the tangent plane parallel to the faces of the slice. 35

3979. (J. J. Sylvester, F.R.S.)—If from the angles of a tetrahedron lines be drawn parallel to those joining the centroids of the opposite faces with (O) any fixed point in space, prove (1) that they will meet in a point I lying in the line (produced) which passes through the fixed point and the centroid (G) of the tetrahedron, and (2) that in the line OGI so obtained GI = 3OG. 38

3988. (C. Taylor, M.A.)—Two conics being inscribed in a quadrilateral, a variable conic has double contact with both of them; find the locus of intersection of the variable common tangents. 101

3989. (G. O'Hanlon.)—If a quadrilateral has the sum of the squares of the diagonals equal to the sum of the squares of the sides, prove that it is a plane figure and a parallelogram. 55

3992. (A. M. Nash.)—S is a focus of a conic, P any point on the curve; with centre P, and radius equal to $SP - k$, k being any constant, a circle is drawn; show that its envelope consists of a pair of circles whose centres are at the foci, and whose radii are k and $2a \mp k$, where $2a$ is the major axis, the upper sign applying to the ellipse and the lower to the hyperbola; and that in the parabola the latter circle becomes a straight line parallel to the directrix and at a distance k from it..... 107

3993. (M. Collins, B.A.)—If A, B, C be the vertices of a hypocycloid of three branches described by a point on the circumference of a circle which rolls within a fixed circle whose diameter is three times that of the rolling circle; and if A'BC be an equi-

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lateral triangle, A′ and A being upon opposite sides of BC, P being any point in the hypocycloid, PE perpendicular to BC, and PD a tangent to the circle whose centre is A′ and radius is A′B or A′C; prove that $PD^4 \div PE^3$ will be equal to sixteen times the diameter of the rolling circle. 110

3997. (W. C. Otter, F.R.A.S.)—Suppose a man has a calf which at the end of three years begins to breed, and afterwards brings forth a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow calf every year; and that these last breed in the same manner, &c. Required the owner's stock at the end of x years. 106

3998. (H. Murphy.)—If on the middle segment of a trisected line a circle be drawn, and if any point on its circumference be joined to the four points on the trisected line, prove that the product of the tangents of the extreme angles at this point is equal to $\frac{1}{4}$. 81

4001. (Francis Galton, F.R.G.S.)—A large nation, of whom we will only concern ourselves with the adult males, N in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, a_0 per cent. of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and so on, up to a_5 who have five. Find (1) what proportion of the surnames will have become extinct after r generations; and (2) how many instances there will be of the same surname being held by m persons. 103

4003. (Rev. Dr. Booth, F.R.S.)—Two planes are drawn parallel to the circular sections of an oblate spheroid at the distance bs^{-1} from the centre; and a secant plane is drawn, cutting the surface in a conic section, and the two parallel planes in two straight lines. Show that the diametral planes through these straight lines are parallel to the circular sections of the cone, whose vertex is the centre and base the conic section in which the surface is cut by the secant plane. 53

4007. (The Editor.)—Show that the equation whose roots are the differences of the roots of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

is $$x^{12} - 48\,Hx^{10} + 8\,(96\,H^2 + I)\,x^8 - 32\,(128\,H^3 + 16\,HI + 13\,J)\,x^6 + 16\,(384\,H^2I + 288\,HJ - 7\,I^2)\,x^4 - 1152\,(2\,HI^2 + 3\,IJ)\,x^2 + 256\,(I^3 - 27\,J^2) = 0,$$

where

$$H = b^2 - ac,\ I = ae - 4bd + 3c^2,\ J = ace + 2bcd - ad^2 - eb^2 - c^3 \ldots$$ 52

4010. (Professor Clifford.)—Prove that the lines of curvature of a quadric surface are projected from an umbilic on a plane parallel to its tangent plane into a series of confocal Cartesian ovals. 73

4018. (B. Williamson, M.A.)—If the equation of a curve be of the form $r^2 - a^2 = mp^2$; prove that the equation of its evolute is $r^2 - (1-m)\,a^2 = mp^2$. 54

4023. (W. H. H. Hudson, M.A.)—A series of strings knotted in squares like a rope-ladder form a belt round a sphere of radius

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r; if $r\theta$ be the length of a side, n the number of squares to reach right round, prove that $\tan\frac{\theta}{2} = \sin\frac{\pi}{n}$. 68

4032. (Professor Crofton, F.R.S.)—A random shot has struck a circular target (area Ω); the chance that it has fallen within a specified space on the target of any form (area ω) is of course $p = \frac{\omega}{\Omega}$; but if another random shot has struck the target *further* from the centre than the first, show that this additional knowledge alters the above probability to $p_1 = 2\frac{\omega}{\Omega} - \frac{m}{M}$, where m, M are the moments of inertia of the areas ω, Ω round the centre of Ω. If the radii of gyration are equal, p is not altered. Also find p if the second shot be *nearer* instead of further. 57

4034. (Professor Clifford.)—Prove that the forty umbilics of a cubic surface which passes once, or of a quartic surface which passes twice, through the imaginary circle at infinity, lie by fives upon sixteen straight lines. 77

4039. (C. H. Hinton.)—On the portions of the sides of a triangle enclosed between the internal and external bisectors of the opposite angles respectively as bases, are constructed three rectangles of equal area; show that the side of one is equal to the sides of the other two..... 63

4041. (W. S. Burnside, M.A.)—In the analysis of the fundamental formulæ for the addition and subtraction of elliptic functions, prove geometrically that the following transformations are legitimate; viz., first change k to k^{-1}, and then change sin (amu) into k sin (amu). 72

4042. (Professor Mannheim.)—The tangents at a double point A on a quartic cut the curve again in B and C so that BC is a double tangent. Prove that $\frac{\rho_1\rho_1'}{\rho_2\rho_2'} = \frac{AB^3}{AC^3}$, where ρ_1, ρ_2 are the radii of curvature at A to the branches which touch AB, AC, and ρ_1', ρ_2' are the radii of curvature at B and C. 78

4044. (T. T. Wilkinson, F.R.A.S.)—From the foot of one of the perpendiculars of a triangle draw straight lines to the other two perpendiculars and also to the other two sides, so as to make equal angles with these four lines all on the same side; then prove that the four intersections lie on the same straight line. 72

4045. (J. J. Walker, M.A.)—Find the equations which determine the position of the tangent plane to an ellipsoid such that the triangle whose three corners are the points in which it meets the axes may have the minimum area; and consider in particular the cases of the sphere and the prolate spheroid having the square of each minor axis equal to half the square of the major. 91

4046. (B. Williamson, M.A.)—Prove, geometrically, that the equation to the reciprocal polar of an epicycloid, with respect to the centre of the fixed circle, is of the form $r \sin m\theta = c$. [An *analytical* proof is given in Art. 311 of the new edition of Dr. Salmon's *Higher Curves*.] 79

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4048. (Asher B. Evans, M.A.)—Two uniform beams whose lengths are $2a$ and $2b$ rest wholly within a hemispherical bowl with their ends against each other. If r be the radius of the bowl, ϕ the angle made with the vertical by the radius drawn to the junction of the beams, and the lineal unit of each beam be taken as the unit of weight, prove that the condition of equilibrium is

$$\tan\phi = \frac{a^2(r^2-a^2)^{\frac{1}{2}} - b^2(r^2-b^2)^{\frac{1}{2}}}{r^2(a+b) - a^3 - b^3} \quad \ldots\ldots 102$$

4057. (J. H. Cotterill, M.A.)—A circle, the centre of which is C, rolls on a fixed circle, the centre of which is O; show that the centre of curvature of the curve traced by any point P rigidly connected with the rolling circle, is given by the following construction:—Through the point of contact (T) of the circles draw TK perpendicular to the normal PT, to meet CP produced in the point K, then OK intersects PT produced in the required centre. 76

4062. (R. Tucker, M.A.)—Give geometrical proofs of the tangential equations (in Cartesian coordinates) for the conic sections..... 70

4063. (Professor Townsend, F.R.S.)—One side of a quadrilateral is fixed, the opposite side passes through a fixed point, and the two variable vertices move on fixed lines; construct it so as to have a given or maximum area. 99

4064. (Sir James Cockle, F.R.S.)—Notation and reference being as in Question 3874, let there be given

$$x^4 r = (x^3 + 2xy)\, p - 4y^2 \quad \ldots\ldots (1).$$

Now Boole (p. 220) in effect states that $y = cx^2$ is the singular solution of (1). Test this statement. 67

4072. (The Editor.)—If in the equations

$$v + x + y + z = a, \qquad v^2 + x^2 + y^2 + z^2 = b,$$
$$v^3 + x^3 + y^3 + z^3 = c, \quad v^4 + x^4 + y^4 + z^4 = d,$$

we have the relation $a^3 - 6ab + 8c = 0$; show (1) that the values of v, x, y, z will be one-fourth of the four values of the expression

$$a \pm \left\{ 4b - a^2 \pm 2\,(a^4 - 4a^2b - 4b^2 + 16d)^{\frac{1}{2}} \right\};$$

and hence show (2) that if $a = 20$, $b = 150$, $c = 1250$, $d = 10674$, the values of v, x, y, z will be 1, 2, 8, 9..... 68

4075. (David Trowbridge.)—Prove that

$$\frac{1}{1+nr} = 1 - \frac{nr}{1+r} + \frac{n(n-1)r^2}{(1+r)(1+2r)} - \frac{n(n-1)(n-2)r^3}{(1+r)(1+2r)(1+3r)} + . \quad 90$$

4082. (T. T. Wilkinson, F.R.A.S.)—In any triangle ABC, if O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles respectively, prove that $AO \cdot AO_1 \cdot AO_2 \cdot AO_3 = AB^2 \cdot AC^2$, $BO \cdot BO_1 \cdot BO_2 \cdot BO_3 = BA^2 \cdot BC^2$, $CO \cdot CO_1 \cdot CO_2 \cdot CO_3 = CB^2 \cdot CA^2$. 107

4097. (Professor Clifford.)—If about a prolate conicoid of revolution there be described an octahedron so that its three diagonals pass through a focus; show that they must be at right angles to each other. 100

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3928. (Proposed by J. J. SYLVESTER, F.R.S.)—If a particle solicited by two attractive forces, varying as the inverse squares of the distances from two fixed points, and by a force varying as the direct distance from a third fixed point midway between the two others, start from rest in the direction of a tangent to the hyperbola, of which the two first named points are the foci, prove that it will continue to oscillate in an arc of the hyperbola.

3229. (Proposed by Professor TOWNSEND, F.R.S.)—A material particle, moving freely under the action of two attractive or repulsive forces emanating from fixed centres, and varying inversely as the square of the distance, sets out without initial velocity from any point on the surface of the sphere coaxal with the two centres of force whose centre divides the interval between them externally in the ratio of the absolute forces; determine the orbit it describes.

I. *Solution by* Professor TOWNSEND, F.R.S.

This very elegant extension (3928) of a well known property may be readily proved on the same principle as the particular case of it when the third force is absent; viz., by supposing the particle constrained to oscillate without friction in the hyperbola in question, and showing that, under the initial circumstances supposed, the entire pressure on the curve throughout the motion is nothing. (See WALTON'S *Mechanical Problems*, 2nd edit., pp. 294, 295.)

Denoting by d_1, d_2, d the initial, and by r_1, r_2, r the current distances of the particle from the far and near foci F_1 and F_2 and from the centre C of the curve, by μ_1, μ_2, μ the corresponding absolute forces, by θ_1, θ_2, θ the angles made by r_1, r_2, r with the external direction of the normal at the particle, by ρ the radius of curvature at the same, and by N the entire normal component, estimated outwards, of the three forces acting on it; then since, evidently,

$$N = \frac{\mu_1}{r_1^2}\cos\theta_1 + \frac{\mu_2}{r_2^2}\cos\theta_2 \pm \mu r\cos\theta$$

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according as the force at C is attractive or repulsive, therefore

$$N\rho = \frac{\mu_1}{r_1^2}\rho\cos\theta_1 + \frac{\mu_2}{r_2^2}\rho\cos\theta_2 \pm \mu r\rho\cos\theta,$$

and since, by the geometry of the hyperbola,

$$\rho\cos\theta_1 = -\rho\cos\theta_2 = \frac{r_1r_2}{a}, \text{ and } \rho\cos\theta = \frac{r_1r_2}{r};$$

therefore

$$N\rho = \left(\frac{\mu_1}{r_1^2} - \frac{\mu_2}{r_2^2}\right)\frac{r_1r_2}{a} \pm \mu r_1r_2,$$

which by hypothesis is initially = 0. Subtracting therefore from its current its evanescent initial value $\left(\frac{\mu_1}{d_1^2} - \frac{\mu_2}{d_2^2}\right)\frac{d_1d_2}{a} \pm \mu d_1d_2$ there remains finally for its current value

$$N\rho = \frac{\mu_1}{a}\left(\frac{r_2}{r_1} - \frac{d_2}{d_1}\right) - \frac{\mu_2}{a}\left(\frac{r_1}{r_2} - \frac{d_1}{d_2}\right) \pm \mu(r_1r_2 - d_1d_2),$$

which, on the hypothesis of no initial velocity, is to be shown, for all values of μ_1, μ_2, μ, to be equal in magnitude and opposite in sign to the square of the current velocity v of the particle in the curve.

But since manifestly, on that hypothesis,

$$v^2 = 2\mu_1\left(\frac{1}{r_1} - \frac{1}{d_1}\right) + 2\mu_2\left(\frac{1}{r_2} - \frac{1}{d_2}\right) \mp \mu(r^2 - d^2),$$

and since, by the geometry of the hyperbola, $r_2 = r_1 - 2a$, $d_2 = d_1 - 2a$, $r_1r_2 = r^2 - (a^2 - b^2)$, $d_1d_2 = d^2 - (a^2 - b^2)$, therefore, &c.

Question 3929 may be answered immediately from the above, from which it appears that, when $\mu = 0$, if initially $d_1^2 : d_2^2 = \mu_1 : \mu_2$, the particle will oscillate freely, if the forces be attractive, in the arc of the hyperbola, of which F_1 and F_2 are the foci, which is terminated by the initial position of the particle and its reflexion with respect to the axis, or describe freely, if they be repulsive, the arc of the same branch between the initial position and infinity. But, by elementary geometry the locus of points for which $d_1^2 : d_2^2 = \mu_1 : \mu_2$ is the sphere, coaxal with F_1 and F_2, whose centre O divides externally the interval F_1F_2 between them so that $OF_1 : OF_2 = \mu_1 : \mu_2$, and the square of whose radius $OR = OF_1 . OF_2$; and therefore, &c.

II. *Solution by* Professor WOLSTENHOLME.

This is true if the particle be started from any point with the proper velocity. Since the particle could describe the hyperbola under any one of the three forces (with negative kinetic energy in the case of two of them) it can describe the hyperbola under the action of all three, provided that (v being the velocity)

$$v^2 = \frac{\mu}{a}\frac{r'}{r} - \frac{\mu'}{a}\frac{r}{r'} - \mu''rr',$$

r, r' being the distances from the near and far foci, and $r' - r = 2a$.

If then the particle start from any point whose distances from the foci are r, r' with such a velocity, and in a direction bisecting the angle between the focal distances it will continue to move in the hyperbola, its velocity continually increasing to the vertex, and then diminishing to zero; after which it will oscillate continually over a certain arc of the hyperbola bounded by two points whose distance r from the nearer focus is given

$$\frac{\mu}{r^2} - \frac{\mu'}{(2a+r)^2} = \mu''a.$$

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If then the particle be initially placed at rest at such a point, it will proceed to describe the arc freely.

If $\mu'' = 0$, we have the case of Question 3929, and it appears that if a particle be placed freely at any point of the sphere $\frac{\mu}{r^2} = \frac{\mu'}{r'^2}$, it will oscillate over an arc of an hyperbola of which the centres of force are foci. This sphere is coaxal with the points, and its centre divides the *produced* distance between the points in the ratio of the absolute forces.

III. *Solution by the* PROPOSER.

Let μ, μ', λ be the intensities of the three forces; $2a$ the major axis of the hyperbola of which the foci are the seats of the attractive forces $\frac{\mu}{r^2}$, $\frac{\mu'}{r'^2}$, where $r - r' = 2a$, so that the total forces along the focal distances are $\frac{\mu}{r^2} + \lambda r$, $\frac{\mu'}{r'^2} + \lambda r'$.

Let the particle be supposed *constrained* to move on the given hyperbola. Then if v is its velocity corresponding to the focal distances, we have

$$v^2 = -\frac{2\mu}{r} - \frac{2\mu'}{r'} + \lambda r^2 + \lambda r'^2 + C$$

$$= \lambda (r - r')^2 + 2\lambda r r' - \frac{\mu}{a}\left(\frac{r - r'}{r}\right) - \frac{\mu'}{a'}\left(\frac{r - r'}{r}\right) \quad 2\lambda r r' + \frac{\mu}{a}\frac{r'}{r} - \frac{\mu'}{a}\frac{r}{r'};$$

the constant being zero since by hypothesis when $v = 0$ $\frac{\mu}{r^2} + \lambda r = \frac{\mu'}{r'^2} + \lambda r$;

or, which is the same thing, $\frac{\mu r'}{r} - \frac{\mu' r}{r'} + 2a\lambda r r' = 0$.

Again, denoting by ω the semi-angle between r, r' and the force normal to the curve by f, we have $f = \left(\frac{\mu}{r^2} - \frac{\mu'}{r'^2} + \lambda (r - r')\right) \sin \omega$,

and if ρ is the radius of curvature, $\rho = \frac{r r'}{a \sin \omega}$.

Hence $$fr = \frac{\mu}{a}\frac{r'}{r} - \frac{\mu'}{a}\frac{r}{r'} + 2\lambda r r' = v^2.$$

Thus there is no resistance, and the forced curve is identical with the free trajectory. (Compare M. Serret's Note 3, on p. 327 of Vol. ii. of Bertrand's Edition of the *Mécanique Analytique*.

The expression used above for the radius of curvature is immediately obtainable from the consideration that if P and Q be two consecutive points in the conic, ω and $\omega + \delta\omega$ the semi-angles which PQ and the consecutive element make with either focal distance, F and G the foci, we have

$$\delta\omega = \tfrac{1}{2}(\mathrm{PFQ} - \mathrm{PGQ}) = \tfrac{1}{2}\left(\frac{\mathrm{PQ}\sin\omega}{r'} - \frac{\mathrm{PQ}\sin\omega}{r}\right) = \frac{\mathrm{PQ}a\sin\omega}{r r'};$$

or $$\frac{1}{\rho} = \frac{\delta\omega}{\mathrm{PQ}} = \frac{a\sin\omega}{r r'}.$$

[The theorem above given comes under the operation of M. Bonnet's general one concerning partial and compound motions, provided we admit the conception of a body describing a real curve with an

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imaginary velocity, and a negative *vis-viva*. Upon this supposition, the hyperbola in the question may be described about each centre of attraction independently, although convex to two of them; and if the forces resolved in the direction of the two focal distances be equal, the joint *vis-viva* will be zero; so that, according to this extended sense given to M. Bonnet's theorem a body starting from rest solicited equally in these two directions and acted on simultaneously by all three centres of force, will move in an hyperbola; that its path is limited to a *portion* of the curve which it ought to describe about a single centre, does not contradict the theorem, although it exhibits it under an unexpected light.]

3931. (Proposed by J. M. WILSON, M.A.)—Show that, in general, eight spheres (and eight only) can be drawn to touch four planes which intersect so as to form a tetrahedron.

I. *Quaternion Solution by* Dr. HOPKINSON.

Let $\mathrm{S}a_1\rho = 1$, &c., be the planes, then the perpendicular from γ on the first is $a_1^{-1}\mathrm{S}(a_1\gamma - 1)$, and if γ be centre of the sphere touching all four

$$a_1^{-1}\mathrm{S}(a_1\gamma-1) = \pm\mathrm{T}a_2^{-1}\mathrm{S}(a_2\gamma-1)$$
$$= \pm\mathrm{T}a_3^{-1}\mathrm{S}(a_3\gamma-1) = \pm\mathrm{T}a_4^{-1}\mathrm{S}(a_4\gamma-1),$$

eight sets of scalar equations of the first degree, three in each set, to find γ. Each set will give one value of γ, hence in general there are eight centres of such circles.

II. *Solution by* Professor WOLSTENHOLME; C. H. HINTON; *and others.*

If p_1, p_2, p_3, p_4 be the four perpendiculars of the tetrahedron, and x the radius of any sphere touching the four sides, then the tetrahedral co-ordinates of the centre of the sphere are $\frac{x}{p_1}, \frac{x}{p_2}, \frac{x}{p_3}, \frac{x}{p_4}$, where one or two of these quantities may be negative.

There will consequently be eight such spheres, the reciprocals of whose radii are the eight positive values of the expression $\pm\frac{1}{p_1}\pm\frac{1}{p_2}\pm\frac{1}{p^3}\pm\frac{1}{p_4}$.

For a regular or for an equi-facial tetrahedron, three of these centres are at infinity and the spheres therefore infinite.

One of the spheres is properly inscribed, four touch one face and the other three produced, and three touch the produced faces only (say in the external dihedral angle $\widehat{\mathrm{BC}}$ and the internal dihedral angle $\widehat{\mathrm{AD}}$). If a, b, c, d be the areas of the faces, the centre of the inscribed sphere is the centre of inertia of particles a, b, c, d at A, B, C, D; the centre of one of four escribed spheres is the centre of inertia of particles $-a, b, c, d$; and the centre of one of three other escribed spheres is the centre of inertia of particles $a, -b, -c, d$. The six planes bisecting the internal dihedral angles meet in one centre; the six bisecting three internal dihedral angles

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at one corner and the three external angles at the opposite face meet in another centre, and we get four such; the six bisecting the internal dihedral angles at two opposite edges and the external dihedral angles at the four other edges all meet in a centre, and there are three of this kind.

III. *Solution by* C. W. MERRIFIELD, F.R.S.

Let the tetrahedron have volume V, and A, B, C, D for its plane faces, these quantities being taken as positive.

Taking any point O in space and letting fall perpendiculars from it on the four planes, we get, disregarding signs,

$$kA + lB + mC + nD = 3V,$$

where k, l, m, n are the four perpendiculars. If each of these perpendiculars $= r$, we have a sphere touching the four planes, and, disregarding signs,

$$r(A + B + C + D) = 3V.$$

This includes the 16 forms comprised in

$$r(\pm A \pm B \pm C \pm D) = 3V.$$

Now observe that if we select any particular sequence of signs, we cannot change the signs throughout, without changing that of V, which is against convention. Hence the 16 forms reduce to 8, therefore, &c.

The selection is easily effected. If we measure from within outwards, we have, for the inscribed sphere,

$$r(A + B + C + D) = 3V.$$

Now bearing in mind that any three faces of a tetrahedron are together greater than the fourth, it follows that we have four cases of the form

$$r(-A + B + C + D) = 3V,$$

excluding the four cases of $r(A - B - C - D)$.

As to the other six cases,

$$\pm r(A + B - C - D) = 3V, \quad \pm r(A - B + C - D) = 3V,$$
$$\pm r(A - B - C + D) = 3V,$$

each pair only gives one case, which must be such as to make V positive. These, therefore, afford three cases only. This gives eight in all.

We may describe the position of the centre of the sphere, in general terms, as being internal to the major sum of the faces opposite in sign.

[On the subject of this question, see SALMON'S *Geometry of Three Dimensions*, 2nd edit., Art. 219.]

3889. (Proposed by Rev. Dr. BOOTH, F.R.S.)—A right angle moves along a conchoid, one side passes through the pole; find the curve enveloped by the other side.

I. *Solution by* Professor TOWNSEND, F.R.S.

The parallel to the free side of the variable right angle, distant from it

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in the direction of the asymptote by the parameter of the curve, intersecting on the asymptote the side constrained to turn round the pole, and enveloping consequently the parabola of which the pole and asymptote are the focus and tangent at vertex, the free side itself therefore envelopes the parallel to that parabola whose parameter is equal in magnitude and sign to that of the curve. And similarly for an angle of any invariable form as well as for a right angle.

II. *Solution by* T. COTTERILL, M.A.

Let a transversal through the node (the pole) cut the conchoid and its asymptote in the points P, P′, A; then AP = AP′ is constant. Hence the required curve is a parallel to the antipedal of the asymptote to the node or the parabola which has the node for its focus and the asymptote for the tangent at its vertex.

Analytically, $(r \pm k) \cos\theta = a$ is the polar equation to the conchoid, and $(p \pm k) \cos\phi = a$ is the class equation to the parallel to a parabola in well known tangential coordinates. The properties of the curve are given by Mr. Roberts, in the March Number of the *Quarterly Journal*, in a "Note on the parallel curves of conics."

3923. (Proposed by the EDITOR.)—(1) From the corners of a triangle cut three triangles of given areas, so that the remainder may be a triangle; also solve the analogous problems for (2) a square, (3) a polygon, and (4) a tetrahedron; and give a numerical example in each case.

Solution by the PROPOSER.

1. Let Δ be the area of the given triangle; a_1, a_2, a_3 its sides; $e_1\Delta$, $e_2\Delta$, $e_3\Delta$ the triangles cut off from the corners, $e_4\Delta$ the middle remaining triangle; and a_1x_1, a_2x_2, a_3x_3 the distances of the points of section from the vertices of the given triangle, estimated in the same direction around the sides. Thus, x_1, x_2, x_3 are the fractional parts of the sides cut off; e_1, e_2, e_3 the fractional parts of the area cut off; and $e_1+e_2+e_3+e_4 = 1$.

Now $\Delta : e_1\Delta = a_1a_2 : a_1x_1(a_3-a_3x_3)$; and proceeding thus we find the conditions of the problem expressible by the following equations:—

$$x_1(1-x_3) = e_1, \quad x_2(1-x_1) = e_2, \quad x_3(1-x_2) = e_3 \ldots\ldots\ldots\ldots (\alpha).$$

Eliminating x_2 and x_3, we obtain

$$(1-e_3)x_1^2 - (1+e_1-e_2-e_3)x_1 + e_1(1-e_3) = 0 \ldots\ldots\ldots\ldots (\beta),$$

whence, putting for shortness' sake $f^2 = e_4^2 - 4e_1e_2e_3$ $\ldots\ldots\ldots\ldots\ldots (\gamma)$,

we find $$x_1 = \frac{2e_1+e_4 \pm f}{2(1-e_3)} = \frac{e_1+\frac{1}{2}(e_4 \pm f)}{e_1+e_2+e_4} \ldots\ldots\ldots\ldots\ldots\ldots (\delta),$$

with similar values for x_2, x_3.

If we had put the unknown quantities to denote the distances of the points of section from the *middles* of the sides, expressed in parts of the

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halves of those sides, respectively; so that, calling these latter unknowns z, the relation between the x's and z's is

$$x = \tfrac{1}{2}(1+z) \quad \text{or} \quad z = 2x-1;$$

the system of equations would have been

$$(1+z_1)(1-z_3) = 4e_1, \quad (1+z_2)(1-z_1) = 4e_2, \quad (1+z_3)(1-z_2) = 4e_3 \ldots(\epsilon);$$

and the points of section would have been given by

$$z_1 = \frac{e_1 - e_2 \pm f}{1-e_3}, \quad z_2 = \&c., \quad z_3 = \&c. \ldots\ldots\ldots\ldots\ldots\ldots (\zeta).$$

The cuttings may, in general, be made in twelve distinct ways; for the parts e_1, e_2, e_3 may cut off in six different ways from the three corners of the triangle, by interchanging x_1, x_2, x_3 on its sides; and at each interchange there are two sets of values for the x's. If however $e_4^2 = 4e_1e_2e_3$, then f vanishes, the pairs of triangular lines of section coincide, and the problem admits of but six solutions; and if $e_4^2 < 4e_1e_2e_3$, the required cutting cannot be done in any way whatever.

If the parts cut off are equal, $e_1 = e_2 = e_3$ ($= e$ say), then from (ζ) we have

$$z_1 = z_2 = z_3 = \frac{\pm f}{1-e} = \pm \frac{\left\{(1-3e)^2 - 4e^3\right\}^{\frac{1}{2}}}{1-e} \ldots\ldots\ldots\ldots\ldots (\eta).$$

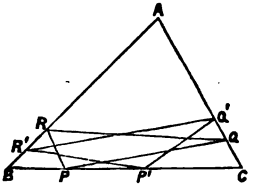
If $e_1 = e_2 = e_3 = e_4$, or $e = \frac{1}{4}$, then $f = 0$, and (η) gives $z_1 = z_2 = z_3 = 0$; thus there is only one way of dividing a triangle into four equal parts by another triangle inscribed in it; namely, by joining the middles of the sides.

By adding equations (ϵ) we obtain

$$z_1z_2 + z_2z_3 + z_3z_1 = 4e_4 - 1, \ = 1 \text{ when } e_4 = \tfrac{1}{2} \ldots\ldots\ldots\ldots\ldots (\theta),$$

which is the relation amongst the z's when the inscribed triangle is equal to $e_4\Delta$ or to $\frac{1}{2}\Delta$.

To take a numerical example; suppose the given triangle ABC to contain 16 square feet, and let it be required to cut off from the corners three triangles containing 1, 2, 10 feet, in such a way that the remainder may be a triangle containing 3 square feet. Then we shall have $e_1 = \frac{1}{16}$, $e_2 = \frac{1}{8}$, $e_3 = \frac{5}{8}$, $e_4 = \frac{3}{16}$, $f = \frac{1}{2}$; and thence from (δ) we find $x_1 = \frac{1}{4}$ or $\frac{7}{12}$, &c. whence the two triangular lines of section obtained by cutting off the triangles 1, 2, 10 from the corners B, C, A, respectively are PQR, P′Q′R′ in the figure, where



$$\text{BP} = \tfrac{1}{4}\,\text{BC}, \quad \text{CQ} = \tfrac{1}{8}\,\text{CA}, \quad \text{AR} = \tfrac{3}{4}\,\text{AB},$$
$$\text{BP}' = \tfrac{7}{12}\,\text{BC}, \quad \text{CQ}' = \tfrac{5}{10}\,\text{CA}, \quad \text{AR}' = \tfrac{15}{16}\,\text{AB}.$$

If, in this case, $e_1 = e_2 = e_3 = 1$, we shall have $e_4 = -2$, and equation (β) will be satisfied by *any* value of x_1. The geometrical interpretation of this is that if AP, BQ, CR be any three parallel lines drawn through A, B, C and meeting the opposite sides in P, Q, R, the triangles AQR, BRP, CPQ will be each equal in area to ABC, and PQR will be double of ABC.

2. We proceed next to apply the same method to cut off from the corners of a *square* four triangles of given areas so that the remainder may be a quadrilateral. Let e_1, e_2, e_3, e_4 denote the ratios of the triangles cut off to the triangular half of the square made by a diagonal; then the system

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of equations is

$$x_1(1-x_4)=e_1,\ x_2(1-x_1)=e_2,\ x_3(1-x_2)=e_3,\ x_4(1-x_3)=e_4 \ldots\ldots\ldots (\iota);$$

whence, by eliminating x_1, x_2, x_3 we find

$$(1-e_2-e_3)x_4^2-(1-e_1-e_2-e_3+e_4+e_1e_3-e_2e_4)x_4+e_4(1-e_1-e_2)=0 \ \ldots\ (\kappa).$$

We thus see that this case of the problem admits, in general, of twelve solutions; inasmuch as when one of the parts is cut from any one indifferently of the four corners (which are here all alike, so that all the distinct solutions may be obtained by permuting the other three) we may interchange the remaining parts in six different ways, and at each interchange there are two values of the distances of the points of section from the vertices.

To take an example in numbers, suppose the triangular parts cut off to be $\frac{1}{16}, \frac{1}{8}, \frac{7}{16}, \frac{5}{32}$ of the square, so that the remainder is a quadrilateral whose area is $\frac{7}{32}$ of the square; then, putting $e_1=\frac{1}{8}$, $e_2=\frac{1}{4}$, $e_3=\frac{3}{8}$, $e_4=\frac{7}{16}$, equation (κ) becomes

$$48x_4^2-56x_4+15=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\lambda),$$

whence we obtain $x_4=\frac{3}{4}$ or $\frac{5}{12}$.

The two systems of values are thus found to be

$$\left.\begin{array}{llll} x_1=\frac{1}{2}, & x_2=\frac{1}{2}, & x_3=\frac{3}{4}, & x_4=\frac{3}{4} \\ x_1=\frac{3}{14}, & x_2=\frac{7}{15}, & x_3=\frac{1}{10}, & x_4=\frac{5}{12} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (\mu).$$

If, in this case, $e_1=e_2=e_3=e_4=\frac{1}{4}$, equation (κ) will be satisfied by *any* value of x_4; and the geometrical interpretation of this is that if ABCD be a square, P any point in AB. M the middle point of AB, MQ parallel to PC, MS parallel to PD, meeting BC and AD (one of the two produced) in Q and S, PR parallel to QS, meeting CD produced in R; then each of the triangles ASP, BPQ, CQR, DRS will be one-half of the semi-square, and the area of the quadrilateral PQRS will be zero.

3. To apply the same method to a *polygon* we have only to let $e_1, e_2, \ldots, e_n$ denote the fractional parts of the triangles formed by drawing diagonals from the ends of each pair of consecutive sides; then the system of equations similar to (α) or (ι) will be

$$x_1(1-x_n)=e_1, \quad x_2(1-x_1)=e_2, \quad \ldots\ldots, \quad x_n(1-x_{n-1})=e_n \ldots\ldots (\nu).$$

The system of equations

$$e=x_1(1-x_2)=x_2(1-x_3)=\ldots\ldots=x_n(1-x_1)$$

which is what (ν) would become when $e_1=e_2=\ldots\ldots=e_n=e$, is discussed in the solution of Question 3528 (*Reprint*, Vol. XVII., p. 24).

4. To take an example in space of three dimensions, let it be required to cut off from the corners of a tetrahedron of volume V four tetrahedral pieces of volumes e_1V, e_2V, e_3V, e_4V, so that the remainder may be an octahedron.

Let x_1, x_2, x_3 be estimated as in Art. 1, around the edges of one face, and x_4, x_5, x_6 from the corners of this face towards the opposite vertex of the tetrahedron. Then, by taking, precisely the same as in the plane problem (1), the ratios of the pieces cut off to the whole volume, the conditions of the problem are expressible by the following system of equations:—

$$\left.\begin{array}{ll} x_1(1-x_3)x_4=e_1, & x_2(1-x_1)x_5=e_2, \quad x_3(1-x_2)x_6=e_3, \\ (1-x_4)(1-x_5)(1-x_6)=e_4 & \end{array}\right\} \ldots\ldots (\xi).$$

Now as there are here but four equations amongst six unknowns, two of them, say x_4 and x_5, may have any values less than unity, and then x_6

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is at once determined; thus the system of equations (ξ) becomes identical in form with (α), viz.,

$$x_1(1-x_3) = \frac{e_1}{x_4} = e'_1, \quad x_2(1-x_1) = \frac{e_2}{x_5} = e'_2, \quad x_3(1-x_1) = \frac{e_3}{x_6} = e'_3 \ldots (\pi),$$

and the solution may be completed as in Art. 1.

To take an example in numbers; let the fractional parts of the whole volume cut off be

$$e_1 = \tfrac{1}{32}, \quad e_2 = \tfrac{3}{32}, \quad e_3 = \tfrac{5}{64}, \quad e_4 = \tfrac{7}{64};$$

then supposing $x_4 = \frac{1}{2}$, $x_5 = \frac{3}{4}$, we have $x_6 = \frac{1}{8}$;

also $e'_1 = \frac{1}{16}$, $e'_2 = \frac{1}{8}$, $e'_3 = \frac{5}{8}$, $e'_4 = \frac{3}{16}$, $f' = \frac{1}{8}$;

thus the numbers are exactly the same as in Art. 1, the triangle ABC being the face along whose edges the parts x_1, x_2, x_3 are cut, and x_4, x_5, x_6 being estimated from B, C, A respectively towards the vertex of the tetrahedron which is opposite to the face ABC.

Since there are twelve ways of making the cuttings on the face ABC, and each of the four faces may be taken for this system of cuttings, it is clear that for each assumed value of x_4, x_5 the required parts may be cut off from the tetrahedron in forty-eight different ways.

3279. (Proposed by J. J. Sylvester, F.R.S.)—If a quadratic equation with real coefficients be written down at hazard, show that the probability of its roots being imaginary is $\frac{3}{7}\frac{1}{2} - \frac{1}{12}\log 2$, or ·3727932.

Solution by the Proposer.

We may suppose the coefficients limited to lie between -1 and 1, fractions of any degree of minuteness being admissible.

Suppose $ax^2 + 2bx + c = 0$ to be the equation.

If a, c have opposite signs, the roots cannot be imaginary. The chance of this is $\frac{1}{2}$.

Let us then suppose a, c to have the same signs, say that both are positive. If, when this is the case, the chance of imaginariness is p, the chance will also be p when a, c are both negative, and consequently when they have the same sign, whether positive or negative; and the absolute chance when the signs of a, c are *the same or contrary* will be $\frac{1}{2}p$.

Now to find the value of p (a, c both positive).

Suppose b constant; if $ac > b^2$ the roots are imaginary, a may have any value from 1 to b^2, and the values of c favourable to imaginariness will extend from 1 to $\frac{b^2}{a}$.

Hence the probability of imaginariness will be

$$\int_{b^2}^{1} da \left(1 - \frac{b^2}{a}\right) = \Big[a - b^2 \log a\Big]_{b^2}^{1} = 1 - b^2 + 2b^2 \log b.$$

Now b may have any value from $\frac{1}{2}$ to $-\frac{1}{2}$; hence we have

$$p = \int_{-\frac{1}{2}}^{\frac{1}{2}} db\,(1 - b^2 + 2b^2 \log b) = 2\int_{0}^{\frac{1}{2}} db\,(1 - b^2 + 2b^2 \log b);$$

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Hence the chance of the roots being imaginary is

$$\tfrac{1}{2}p = \tfrac{1}{2}\Big[b - \frac{b^2}{3} - \frac{2b^3}{9} + \frac{2b^3}{3}\log b\Big] = \tfrac{3}{7}\tfrac{1}{2} - \tfrac{1}{1}\tfrac{1}{2}\log 2 = \cdot 3727932\ldots;$$

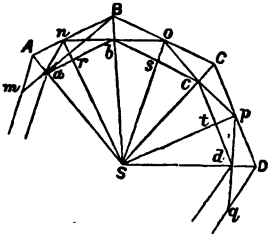
and thus the chance of their being real is ·6272067 ....

[Other Solutions are given on pp. 22—24 of Vol. XV. of the *Reprint*.]

3937. (Proposed by T. T. WILKINSON, F.R.A.S.)—Two similar polygons, whose areas are (A) and (B), are so situated that (B) is enclosed by (A), and their corresponding sides are parallel. If a third polygon (C) be inscribed to (A) and circumscribed to (B), prove that $(C)^2 = (A).(B)$.

Solution by the PROPOSER; A. ESCOTT, M.A.; *and others.*

Let ABCD..., *abcd*..., *mnopq*..., be portions of the polygons (A), (B), (C) respectively; also let S be their common centre of similitude, and join S*n*, S*o*, S*p*,... cutting *ab*, *bc*, *cd*,... in *r*, *s*, *t* ...



Then $\dfrac{Sa}{SA} = \dfrac{Sb}{SB} = \dfrac{Sc}{SC} = \ldots \equiv \lambda$;

therefore $\dfrac{\triangle SaB}{\triangle Sab} = \dfrac{Sanb}{Sab} = \dfrac{SB}{Sb} = \dfrac{1}{\lambda}$,

also $\dfrac{\triangle SaB}{\triangle SAB} = \dfrac{Sa}{SA} = \lambda$;

and the same are true for every part of the system; hence we have

$$\frac{(C)}{(B)} = \frac{1}{\lambda} \text{ and } \frac{(C)}{(A)} = \lambda, \text{ therefore } (C)^2 = (A)(B).$$

[Professor WOLSTENHOLME remarks that there will generally be two such polygons; for if we take *n* any point in AB, and draw *nbo*, *ocp*, &c., we shall arrive at a point *m* on the last side of A; if *mn* pass through *a*, what is required is done; and if not, the points *m*, *n* are corresponding points of two homographic systems, so that *mn* always touches a fixed conic to which, if we draw the two tangents from *a*, the question is solved by either of them. It may, for anything that appears *à priori*, happen that this conic passes through *a*, in which case only one such polygon can be constructed; or it may happen that every line *mn* passes through *a*. It would be interesting to investigate the conditions for this last result.]

3514. (Proposed by the EDITOR.)—Two points are taken at random in each of the opposite sides of a quadrilateral; find the probability that the quadrilateral of which they are the vertices will be less than half the given quadrilateral.

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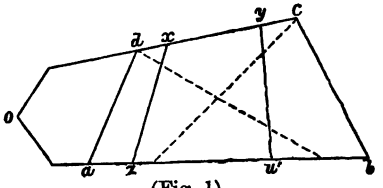
Solution by G. S. CARR.

1. Let two opposite sides meet produced in O; and in this first case let the bisecting lines drawn from the extremities c, d of one of those sides (Fig. 1) cut the opposite side.



(Fig. 1)

Let the distances of the vertices from O be a, b, c, d; and the distances of the four random points x, y, z, u.

Consider x, y, z as fixed; then u may vary from z to w, where w is found from the equation

$$yu - xz = \tfrac{1}{2}(bc - ad) = A, \text{ say}; \quad \text{therefore} \quad w = \frac{A + xz}{y} \quad \ldots\ldots (1).$$

Then as z varies from a to b, u varies from z to w until $w = b$, and then $z = z'$, where $z' = \dfrac{by - A}{x}$ from (1). As z increases from z' to b, u varies always from z to b.

These limits for u and z hold for all values of x between d and c, and of y between x and c.

The probability of any one configuration is $4\,\dfrac{dx\,dy\,dz\,du}{(b-a)^2(c-d)^2}$; for each will occur four times if the two lines are not drawn to cross.

Therefore the probability of the varying quadrilateral being less than half the given one, is

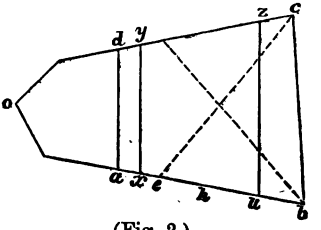
$$p_1 = \frac{4}{(b-a)^2(c-d)^2}\left\{\int_d^c\int_x^c\int_a^z\int_z^w dx\,dy\,dz\,du + \int_d^c\int_x^c\int_{z'}^c\int_z^b dx\,dy\,dz\,du\right\}.$$

Similarly the probability of its being greater than half the given quadrilateral, is $p_2 = \dfrac{4}{(b-a)^2(c-d)^2}\displaystyle\int_d^c\int_x^c\int_a^{z'}\int_{w'}^b dx\,dy\,dz\,du$;

$$\therefore \quad p_1 + p_2 = \frac{4}{(b-a)^2(c-d)^2}\int_d^c\int_x^c\int_a^b\int_z^b dx\,dy\,dz\,du = 1, \text{ or certainty.}$$

The integrations are easily effected, but the results are unwieldy.

2. The only other distinct case that can occur when the sides cd, ab are not parallel, is that in which the lines drawn from b and c to bisect the figure (Fig. 2), cut the opposite sides ab, cd. There is now a point h in ab, where the triangle $dhc = \tfrac{1}{2}abcd$; and a similar point in dc. This peculiarity adds to the intricacy of the problem. Employing the same notation, but altering the order of integration as the letters in the figure show, we shall find for p_2 the following value.



(Fig. 2.)

The integral $\dfrac{4}{(b-a)^2(c-d)^2}\iiiint dx\,dy\,dz\,du$

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is to be taken five times, the limits of the successive integrations being exhibited in the following table, in which the limits are all obtained from the condition $uz - xy > \frac{1}{2}(bc - ad) = A$.

	x	y	z	u
1st Integral	a to e	d to y_1	z_1 to c	u_1 to b
2nd ,,	a to e	y_1 to c	y to c	u_1 to b
3rd ,,	e to h	d to y_2	z_1 to c	u_1 to b
4th ,,	h to b	d to y_3	z_1 to z_2	x to b
5th ,,	h to b	y_3 to y_2	z_1 to c	u_1 to b

$$i.e., \int_a^e \int_d^{y_1} \int_{z_1}^c \int_u^b dx\,dy\,dz\,du$$

&c., &c.; and we have

$$e = \frac{bc + ad}{2c}, \quad h = \frac{A}{c - d};$$

$$u_1 = \frac{xy + A}{z}, \quad y_1 = \frac{A}{b - x};$$

$$z_1 = \frac{xy + A}{b}, \quad y_2 = \frac{bc - A}{x};$$

$$z_2 = \frac{xy + A}{x}, \quad y_3 = \frac{cx - A}{x};$$

3. If ab and cd are parallel, a and d may be taken as the initial points; and the result of the first investigation used, putting $a = d = 0$; and for the equation we have $y - x + u - z = \frac{1}{2}(b + c)$, &c.

3731. (Proposed by S. Watson.)—EG is a focal chord in an ellipse, and MP a perpendicular to it at its middle point M. Show that the curve which the line MP always touches is a three-cusped curvilinear triangle, to which the axes of the ellipse are tangents, and that its area is to that of the ellipse as $e^6 : 1 - e^2$, where e is the excentricity.

Solution by Asaph Hall, M.A.

Taking the centre of the ellipse as the origin and the principal axes for the axes of coordinates, and adopting the usual notation, we have for the equation of the perpendicular,

$$a^2ym^3 + (a^2x - c^3)\,m^2 + b^2ym + b^2x = 0;$$

m being the tangent of the angle that the focal chord makes with the axis of x. The equation of the curve sought will be found by eliminating m between the preceding equation and its first derivative with respect to m. Hence we have

$$4a^2b^4y^4 + (8a^4x^2 + 20a^2c^3x - c^6)\,b^2y^2 + 4x\,(a^2x - c^3)^3 = 0 \quad \ldots\ldots\ldots (1).$$

Now $c = ae$, and if we put $ae^3 = h$, and solve for y, we have

$$y = \pm \frac{a}{8b}\left\{\sqrt{h} \pm \sqrt{(h + 8x)}\right\}^{\frac{1}{2}}\left\{3\sqrt{h} \mp \sqrt{(h + 8x)}\right\}^{\frac{3}{2}}, \quad \ldots\ldots\ldots (2),$$

where the upper and lower signs in the radical expressions must be taken together. The elegant form of y in (2) arises from the fact that in the

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solution of (1) the terms in x retain a cubic form through two successive reductions. From equation (2) we see that the curve is confined to the limits $x = +h$, and $x = -\frac{1}{8}h$, and has double points for these values of x.

If we denote the area of the curve by A, we have

$$\frac{4b}{a}\mathrm{A} = \int_{-\frac{1}{8}h}^{h} \left\{\sqrt{h} + \sqrt{(h+8x)}\right\}^{\frac{1}{2}} \left\{3\sqrt{h} - \sqrt{(h+8x)}\right\}^{\frac{1}{2}} dx$$

$$- \int_{-\frac{1}{8}h}^{0} \left\{\sqrt{h} - \sqrt{(h+8x)}\right\}^{\frac{1}{2}} \left\{3\sqrt{h} + \sqrt{(h+8x)}\right\}^{\frac{1}{2}} dx.$$

Put now $f = \sqrt{h}$, and $z = \sqrt{(h+8x)}$, and we have

$$\frac{16b}{a}\mathrm{A} = \int_{0}^{3f} (f+z)^{\frac{1}{2}} (3f-z)^{\frac{1}{2}} z\,dz - \int_{0}^{f} (f-z)^{\frac{1}{2}} (3f+z)\, z\,dz,$$

$$= \int_{-f}^{2f} (4f^2-z^2)^{\frac{1}{2}} (2f^2-z^2+fz)\,dz - \int_{f}^{2f} (4f^2-z^2)^{\frac{1}{2}} (z^2+fz-2f^2)\,dz.$$

The integrations are now readily performed, and we find

$$\mathrm{A} = \frac{af^2}{8b}\pi = \frac{c^2e}{8b^2}\pi = \frac{c^2}{1-e^2} \text{ (area of ellipse).}$$

[Other Solutions are given in the *Reprint*, Vol. XVII., p. 105, where a figure of the curve is drawn.]

3882. (Proposed by Professor Townsend, F.R.S.)—The circumscribed and inscribed circles of a variable triangle, plane or spherical, being supposed both fixed; show that, throughout the deformation of the triangle, velocity of A : velocity of B : velocity of C $= \cot\frac{1}{2}$A : $\cot\frac{1}{2}$B : $\cot\frac{1}{2}$C in either case; and hence that, angular velocity of a : angular velocity of b : angular velocity of $c = a : b : c$ for the plane, and $= \tan\frac{1}{2}a : \tan\frac{1}{2}b : \tan\frac{1}{2}c$ for the spherical triangle.

Solution by the Proposer.

For, evidently,

velocity of A : velocity of B : velocity of C $= (s-a) : (s-b) : (s-c)$,

or $= \sin(s-a) : \sin(s-b) : \sin(s-c)$

according as the triangle is plane or spherical, and therefore &c., as regards the first part; from which the second follows, for the sphere immediately by virtue of the polar triangle, and for the plane either indirectly as the limit for the sphere, or directly from the evident property that the angular velocity of any side of the variable triangle is equal to half the sum of the angular velocities of its extremities round the centre of the circle.

3932. (Proposed by the Editor.)—From a bag containing n marbles a number is taken at random and put into another bag; show (1) that if

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a number be taken at random from the second bag, the respective probabilities of obtaining an odd and an even number are as

$$\Sigma_{r=1}^{r=n} \frac{C(n,r)}{2^r-1} + 2^n - 1 : 2^n - 1 - \Sigma_{r=1}^{r=n} \frac{C(n,r)}{2^r-1},$$

where $C(n,r)$ denotes the number of combinations of n things taken r at a time; and hence show (2) that if $n=12$, these probabilities are as

8687255531747712 : 80838947728038263.

Solution by Asher B. Evans, M.A.; A. Martin; A. B. Escott, M.A.; *and others.*

1. Let p_n and q_n be the respective probabilities of obtaining an odd and an even number. The probability that r marbles will be taken from the first bag and placed in the second is $\frac{C(n,r)}{2^n-1}$; and the probability that an *odd* number will be drawn from the second bag, on the supposition that it contains r marbles, is $\frac{2^{r-1}}{2^r-1}$; also $\frac{2^{r-1}-1}{2^r-1}$ is the probability that an *even* number will be drawn, on the same supposition. We have, therefore,

$$p_n : q_n = \Sigma_{r=1}^{r=n} \left\{ \frac{C(n,r)}{2^n-1} \cdot \frac{2^{r-1}}{2^r-1} \right\} : \Sigma_{r=1}^{r=n} \left\{ \frac{C(n,r)}{2^n-1} \cdot \frac{2^{r-1}-1}{2^r-1} \right\}.$$

Multiplying the second couplet by $2(2^n-1)$, and observing that

$$\frac{2^r}{2^r-1} = 1 + \frac{1}{2^r-1}, \quad \text{and} \quad \frac{2^r-2}{2^r-1} = 1 - \frac{1}{2^r-1},$$

we have, since $\Sigma_{r=1}^{r=n} C(n,r) = 2^n-1$,

$$p_n : q_n = \Sigma_{r=1}^{r=n} \frac{C(n,r)}{2^r-1} + 2^n - 1 : 2^n - 1 - \Sigma_{r=1}^{r=n} \frac{C(n,r)}{2^r-1} \quad \ldots\ldots\ldots (1).$$

2. By putting $n=12$ in (1), and clearing of fractions, we find

$$p_{12} : q_{12} = 8687255531747712 : 80838947728038263.$$

3918. (Proposed by the Rev. W. Roberts, M.A.)—Let a polygon be inscribed in an equilateral hyperbola: prove that the algebraic sum of each side, multiplied by the distance of its middle point from the centre if equal to zero.

I. *Solution by* A. M. Nash.

Let $x^2-y^2=a^2$ be the equation of the hyperbola; C its centre; (x_1, y_1), (x_2, y_2), ... (x_n, y_n) the coordinates of the vertices $P_1, P_2, \ldots P_n$ of the poly-

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gon; and A_{12}, $A_{2,3}$, ... the middle points of the sides P_1P_2, P_2P_3, &c. Then

$$(P_r, P_{r+1})^2 . (CA_{r,r+1})^2$$
$$= \left\{(x_{r+1}-x_r)^2 + (y_{r+1}-y_r)^2\right\}\left\{\left(\frac{x_r+x_{r+1}}{2}\right)^2 + \left(\frac{y_r+y_{r+1}}{2}\right)^2\right\}$$
$$= (x_{r+1}y_{r+1} - x_r y_r)^2;$$ whence we obtain

$$\Sigma(P_rP_{r+1} . CA_{r,r+1}) = (x_2y_2 - x_1y_1) + (x_3y_3 - x_2y_2) \ldots\ldots + (x_1y_1 - x_ny_n) = 0.$$

II. *Solution by* Professor TOWNSEND, F.R.S.; G. EDMUNDSON; *and others.*

The equation of the curve referred to its rectangular asymptotes being $xy = a^2$; let (x_1, y_1), (x_2, y_2), (x_3, y_3), ... (x_n, y_n) be the coordinates of the several vertices, in order of sequence, of any inscribed polygon $P_1P_2P_3 \ldots P_n$; then since the doubles of the several products in question, for the several sides P_1P_2, P_2P_3, &c., P_nP_1, are respectively

$$[x_1^2 - y_1^2] - [x_2^2 - y_2^2], \quad [x_2^2 - y_2^2] - [x_3^2 - y_3^2], \ldots\ldots [x_n^2 - y_n^2] - [x_1^2 - y_1^2],$$

whose sum evidently = 0, therefore, &c.

1987. (Proposed by Dr. BALL.)—If α, β, γ, δ be the roots of $(a, b, c, d, e)(x, 1)^4 = 0$, and α', β', γ', δ' the roots of $(a_1, b_1, c_1, d_1, e_1)(x, 1)^4 = 0$, find the equation which has for its roots all the different values of

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix}$$

Solution by the PROPOSER.

Representing by I, J, I_1, J_1 the usual invariants, by λ, μ, ν the roots of $a^3x^3 - aIx + 2J = 0$, and by λ', μ', ν' the roots of $a_1^3x^3 - a_1I_1x + 2J_1 = 0$, the determinant reduces to

$$4\begin{vmatrix} \lambda, & \lambda', & 1 \\ \mu, & \mu', & 1 \\ \nu, & \nu', & 1 \end{vmatrix}$$

Denoting the latter expression by $4z$, and substituting the values for λ, μ, ν, &c., found by solving $a^3x^3 - aIx + zJ = 0$, we deduce the following equation for z:

$$a^{12}a_1^{12}z^{12} - 4a^{10}a_1^{10}\Pi_1 z^{10} + 6a^8a_1^8I^2I_1^2z^8$$
$$+ a^6a_1^6(432J^2J_1^2 - 8I^3J_1^3 - 8J^2I_1^3 - 4I^3I_1^3)z^6$$
$$+ a^4a_1^4(I^4I_1^4 - 864J^2J_1^2 + 16I^3J_1^2 + 16J^2I_1^3)z^4$$
$$+ a^2a_1^2(432I^2J^2I_1^2J_1^2 - 8I^5I_1^2J_1^2 - 8I^2J^2I_1^5)z^2$$
$$+ 16(J^4I_1^6 + I^6J_1^4 - 2I^3J^2I_1^3J_1^3) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

[The determinant in the Question, when equated to zero, is the condition that one of the biquadratics can be changed by a linear transformation into the other.

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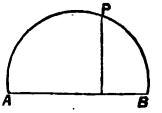
The continued product of the twelve roots of the resulting equation (A) only differs by a numerical factor from $(I_1^3J^2 - I^3J_1^2)^2$. This might have been predicted, since if one of the biquadratics could have been changed into the other by a linear transformation, we must have $\frac{I^3}{I_1^3} = \frac{J^2}{J_1^2}$.]

3662. (Proposed by G. S. Carr.)—A rectangular lamina of length a, width b, and thickness c, elastic only to flexure in a plane perpendicular to its width, is to be cut so that when A, B, the middle points of its ends, are attached to a string of length $d < a$, it may form the cylindrical surface whose equation, with A for origin and AB for axis of x, is $y = f(x)$. Show generally how to determine its shape; first, when the width only is varied, and b is its maximum value; secondly, when the thickness only is altered, and c, which is but small, is allowed for its maximum value.

Solution by the Proposer.



Let P be a point on the curve $y = f(x)$; let x, y be the coordinates; ρ the radius of curvature; w, u the width and thickness of lamina; I the moment of inertia of cross section of lamina about the neutral axis; s the curve length AP. Also let E be the modulus of elasticity, and T the tension of the string.

Then the moment of exterior forces at P $= Ty$, and the moment of the interior forces $= \frac{EI}{\rho} = \frac{Ewu^3}{12\rho}$; therefore the equation of equilibrium is

$$\frac{E}{12T} wu^3 = \rho y = F(x) \quad \ldots\ldots (1),$$

a known function of x, since $y = f(x)$, and $\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \div \frac{d^2y}{dx^2}$.

First let $u = c$ throughout. Then since w, when a maximum, $= b$, the corresponding value of x is determined by the condition $F'(x) = 0$. Let x' be that value; therefore from (1) we have $\frac{EC^3}{12T} = \frac{F(x')}{b}$;

therefore by substituting in (1) we obtain $\frac{w}{b} = \frac{F(x)}{F(x')} \quad \ldots\ldots (2).$

Also $$s = \int_0^x \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \quad \ldots\ldots (3).$$

By eliminating x between (2) and (3), the equation in w and x is obtained, which determines the figure of the lamina.

Secondly, let $w = b$ throughout; then $u = c$ when $x = x'$, and in the same manner as before, we find for equation (2) $\frac{u^3}{v^3} = \frac{F(x)}{F(x')}$.

Eliminate x again between (2) and (3), and the resulting equation in u and z determines the form of the section of the lamina when its width is invariable.

It appears from the above that the figure of the lamina is independent of the forces in action, if the elasticity be the same throughout.

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3960. (Proposed by the EDITOR.)—From a point P in the plane of a triangle ABC perpendiculars PD, PE, PF are drawn on the sides; prove (1) that the average area of the triangle DEF is one-eighth of the triangle ABC if P be allowed to range over the circle drawn round ABC, but (2) if P be limited to points within the triangle ABC, the average is

$$1\tfrac{1}{2}(\sin^2 A + \sin^2 B + \sin^2 C)\,\Delta, \quad \text{or} \quad \tfrac{1}{8}(1 + \cos A \cos B \cos C)\,\Delta,$$

$$\text{or} \quad \frac{1}{12}\left(\frac{a^2+b^2+c^2}{4R^2}\right)\Delta.$$

where Δ is the area of the triangle, and R the radius of the circumscribed circle.

Solution by the PROPOSER.

1. Let k be the distance of P from the centre of the circumscribed circle, and R the radius of this circle; then in my Quest. 1279 (*Reprint*, Vol. XVIII., p. 95) it is shown that the area of the triangle DEF is $\frac{1}{4}\left(1 - \frac{k}{R^2}\right)\Delta$; hence the average required in this first part of the question is

$$\frac{1}{\pi R^2}\int_0^R \frac{1}{4}\left(1 - \frac{k^2}{R^2}\right)\Delta \cdot 2\pi k\,dk = \frac{\Delta}{4R^2}\int_0^R \left(1 - \frac{x}{R^2}\right)dx = \frac{\Delta}{8}.$$

2. Taking A as origin, and AB and AC as axes of x and y, the area of the triangle PDE is $\frac{1}{2}xy\sin^2 A$; hence when P ranges over the whole triangle ABC, the average of the triangle PDF is

$$\frac{1}{\Delta}\int_0^c\int_0^{b\left(1-\frac{x}{c}\right)} \tfrac{1}{2}xy\sin^2 A \cdot dx\,dy \sin A = \frac{b^2\sin^4 A}{4\Delta}\int_0^c \left(1 - \frac{x}{c}\right)^2 x\,dx$$

$$= \frac{b^2c^2\sin^4 A}{4\Delta}\int_0^1 (1-z)\,z^2\,dz = \frac{b^2c^2\sin^4 A}{48\Delta} = \tfrac{1}{12}\Delta\sin^2 A.$$

Hence the average area of the triangle DEF is at once seen to have the first of the values assigned to it in the question, and it can also be expressed in either of the other two equivalent forms.

3974. (Proposed by the Rev. R. HARLEY, F.R.S.)—Required (1) an explicit solution of the equation

$$(1 + ax^2)\frac{d^2y}{dx^2} + ax\frac{dy}{dx} - n^2y = 0;$$

and (2) deduce therefrom the solution of the equation

$$x^2(x^2 + a)\frac{d^2y}{dx^2} + x(2x^2 + a)\frac{dy}{dx} - n^2y = 0.$$

I. *Solution by the* PROPOSER.

The solution of the original equation is usually effected by changing the independent variable.

Assume $t = \int^x \frac{dx}{(1+ax^2)^{\frac{1}{2}}}$, then the equation reduces to $\frac{d^2y}{dt^2} = n^2y$,

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a known integral form. We have, in fact, $2\frac{dy}{dt}\frac{d^2y}{dt^2} = 2n^2y\frac{dy}{dt}$,

and integrating, $\left(\frac{dy}{dt}\right)^2 = n^2y + c$,

whence $t = \int \frac{dy}{(n^2y^2+c)^{\frac{1}{2}}} = \frac{1}{n}\log\left\{y + \frac{1}{n}(n^2y^2+c)^{\frac{1}{2}}\right\} + c'$.

But $t = \int \frac{dx}{(1+ax^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}\log\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\} + c''$.

Whence, comparing these values of t, effecting some obvious reductions, and modifying constants, we obtain, finally,

$$y = k_1\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{\frac{n}{\sqrt{a}}} + k_2\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{-\frac{n}{\sqrt{a}}}.$$

And the solution of the second equation is derived from this by simply writing x^{-1} in place of x. We thus have

$$y = k_1\left\{\frac{\sqrt{a}}{x} + \frac{1}{x}(x^2+a)^{\frac{1}{2}}\right\}^{\frac{n}{\sqrt{a}}} + k_2\left\{\frac{\sqrt{a}}{x} + \frac{1}{x}(x^2+a)^{\frac{1}{2}}\right\}^{-\frac{n}{\sqrt{a}}}.$$

The former result coincides with that obtained by a different method in Art. 22 of my Paper, entitled "On the Theory of the Transcendental Solution of Algebraic Equations," published in Vol. V. of the *Quarterly Journal of Mathematics*. The latter result is there wrongly given; it should have been written as above.

I here reproduce in a fuller form the solution indicated in the Article above referred to.

The symbolical expression $(1+ax^2)\frac{d^2}{dx^2} + ax\frac{d}{dx} - n^2$,

is decomposable into the factors

$$(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} + n \quad \text{and} \quad (1+ax^2)^{\frac{1}{2}}\frac{d}{dx} - n,$$

the order in which the factors are taken being, in this case, indifferent. It is to be observed that these factors are *symbolical* as distinguished from *algebraical*, and that they are to be dealt with according to the laws of the *calculus of operations*.

We have now, successively,

$$\left\{(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} + n\right\}\left\{(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} - n\right\}y = 0;$$

$$\left\{(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} - n\right\}y = \left\{(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} + n\right\}^{-1}0$$

$$= k\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{-\frac{n}{\sqrt{a}}};$$

$$y - \left\{(1+ax^2)^{\frac{1}{2}}\frac{d}{dx} - n\right\}^{-1}\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{-\frac{n}{\sqrt{a}}}$$

$$= k_1\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{\frac{n}{\sqrt{a}}} + k_2\left\{x\sqrt{a} + (1+ax^2)^{\frac{1}{2}}\right\}^{-\frac{n}{\sqrt{a}}}.$$

The proposed equations derive a special interest from their connection with the theory of "Differential Resolvents."

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[In the particular case $a=-1$, $n=\frac{1}{3}$, the equations in the question become

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+\frac{1}{3^2}y=0, \quad x^2(x^2-1)\frac{d^2y}{dx^2}+x(2x^2-1)\frac{dy}{dx}+\frac{1}{3^2}y=0,$$

which are the respective differential resolvents of the equations

$$y^3-3y+2x=0, \quad y^3-3y+\frac{2}{x}=0.$$

For information as to how these differential resolvents are formed, and what is their connexion with the theory of algebraic equations, we would refer to Mr. HARLEY's above-cited paper. Moreover, the principles of the method of symbolical decomposition and solution employed here by Mr. HARLEY are very briefly and clearly explained in Arts. 21—29 of a paper by Sir J. COCKLE, in Vol. I. (pp. 164—168) of the *Messenger of Mathematics*.]

II. *Solution by* Professor WOLSTENHOLME, M.A.

$$(1+ax^2)\frac{d^2y}{dx^2}+ax\frac{dy}{dx}-n^2y=0, \quad \frac{d}{dx}\left\{(1+ax^2)^{\frac{1}{2}}\frac{dy}{dx}\right\}=\frac{n^2y}{(1+ax^2)^{\frac{1}{2}}};$$

therefore

$$\left\{(1+ax^2)^{\frac{1}{2}}\frac{dy}{dx}\right\}^2=n^2y^2+b^2,$$

$$\frac{1}{n}\log\frac{\{ny+(n^2y^2+b^2)^{\frac{1}{2}}\}}{C}=\frac{1}{a^{\frac{1}{2}}}\log\left\{a^{\frac{1}{2}}x+(1+ax^2)^{\frac{1}{2}}\right\};$$

whence

$$y=A\left\{(1+ax^2)^{\frac{1}{2}}+a^{\frac{1}{2}}x\right\}^{\frac{n}{\sqrt{a}}}+B\left\{(1+ax^2)^{\frac{1}{2}}-a^{\frac{1}{2}}x\right\}^{\frac{n}{\sqrt{a}}}.$$

Also, putting $\frac{1}{x}$ for x in the original equation, it becomes

$$x^2(x^2+a)\frac{d^2y}{dx^2}+x(2x^2+a)\frac{dy}{dx}-n^2y=0,$$

of which the solution is therefore

$$yx^{\frac{n}{\sqrt{a}}}=A\left\{(x^2+a)^{\frac{1}{2}}-a^{\frac{1}{2}}\right\}^{\frac{n}{\sqrt{a}}}+B\left\{(x^2+a)^{\frac{1}{2}}-a^{\frac{1}{2}}\right\}^{\frac{n}{\sqrt{a}}}.$$

3978. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Prove that the volume of any slice of a surface of the second order, bounded by parallel planes, may be found by the formula $V=\frac{1}{2}t(A+B)+\left(\frac{abc}{p^3}\right)\frac{1}{6}(t^3\pi)$, where A and B are the areas of the parallel slices, t the thickness of the slice, and p the perpendicular from the centre on the tangent plane parallel to the faces of the slice.

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Solution by G. S. CARR.

Let $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ be the equation to a conicoid, the y and z axes being taken at right angles, their plane parallel to the given cutting planes, and the x axis the conjugate diameter. Let h, k be the intercepts on the x axis of the two bounding planes. Then, writing m for $\frac{\pi\beta\gamma}{\alpha^2}$, we shall have $\quad A=m(\alpha^2-h^2),\quad B=m(\alpha^2-k^2),$

$$V=\int_k^h m(\alpha^2-x^2)\frac{p}{\alpha}\,dx=\frac{p}{\alpha}\left\{m\alpha^2(h-k)-\tfrac{1}{3}m(h^3-k^3)\right\}\ldots\ldots(1).$$

Now $\quad A+B=m(2\alpha^2-h^2-k^2);$

therefore $\quad m(h^2+k^2)=2m\alpha^2-A-B\ldots\ldots(2).$

Again $\quad \tfrac{1}{3}m(h^3-k^3)=\tfrac{1}{3}m(h-k)(h^2+k^2)-\tfrac{1}{3}m(h-k)^3$

$$=m\alpha^2(h-k)-\tfrac{1}{3}(A+B)(h-k)-\tfrac{1}{3}m(h-k)^3, \text{ by (2).}$$

Substitute this in (1), thus we obtain

$$V=\frac{p}{\alpha}\left\{\tfrac{1}{3}(A+B)(h-k)+\tfrac{1}{3}m(h-k)^3\right\}=\tfrac{1}{3}(A+B)t+\frac{\pi abc}{p^3}\frac{t^3}{6},$$

since $\pi\beta\gamma p=\frac{1}{4}$ volume of conicoid $=\pi abc$, and $\frac{h-k}{t}=\frac{\alpha}{p}$.

3957. (Proposed by Professor TOWNSEND, F.R.S.)—Show that a material particle, moving freely in the plane of a thin uniform circular ring, whose elements attract it according to the law of the inverse cube of the distance, will, if projected with the velocity just sufficient to carry it to infinity against the action of the resultant force, describe the arc of the epicycloid (or hypocycloid) concentric with the ring, which commences at the point in the direction of projection, and terminates at the first cusp on the circumference of the circle.

I. *Solution by the* PROPOSER.

From the known relation between the central radius vector r and corresponding perpendicular p on tangent in any epicycloid (or hypocycloid if b be negative), viz., $\quad p^2=\frac{(a+2b)^2}{4b(a+b)}(r^2-a^2),$

where a and b are the radii of the fixed and rolling circles respectively, and from the known formula for the force F in any orbit described freely round any fixed centre, viz., $\quad F\,dr=\tfrac{1}{2}h^2\,d(p^{-2}),$

where h is the elementary or synchronous area, it follows immediately that in any epicycloid (or hypocycloid), the centre of force being that of the fixed circle, $\quad F=-h^2\cdot\frac{4b(a+b)}{(a+2b)^2}\cdot\frac{r}{(r^2-a^2)^2}.$

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which, compared with the attraction of a thin uniform circular ring of mass m and radius a for the law of the inverse cube of the distance, viz.,

$$F = \mp m \cdot \frac{r}{(r^2-a^2)^2},$$

according as r is $>$ or $<a$, gives (since $h^2 = v^2p^2$) for the square of the velocity in the orbit under the action of the ring

$$v^2 = \frac{\pm m}{r^2-a^2},$$

which, being that of the velocity which would be acquired in falling in from infinity to the distance r under the action of m, therefore &c.

NOTE.—When the perpendicular through the point to the direction of projection intersects the ring at either coincident or imaginary points, the radius b of the rolling circle (which, when the points of intersection are real, generates the epicycloid) being then either infinite or imaginary, the orbit consequently changes character, and becomes instead a unicuspidal spiral having its cusp on the ring, whose two branches (symmetrical with respect to the cuspidal vector) run into each other from opposite directions (after an infinite number of volutions) at infinity. The involute of the circle, which it becomes for the case of coincidence, is of course familiar to all; but the general spiral for imaginary intersection, whose polar equation (involving logarithmic instead of circular functions) may be readily obtained by integration from the differential relation $p^2 = m^2(r^2-a^2)$ for the case of $m<1$, has not yet, so far as I am aware, been either named or noticed by geometers.

[The intrinsic equation is, of course, $s = a(\epsilon^{m\phi} - \epsilon^{-m\phi})$.]

II. *Solution by* Professor WOLSTENHOLME.

The attraction of the ring on a particle at a distance x

$$= 2\mu \int_0^\pi \frac{a d\phi\,(x - a\cos\phi)}{(x^2+a^2-2ax\cos\phi)^2}$$

$$= \frac{\mu a}{x} \int_0^\pi \left(\frac{1}{x^2+a^2-2ax\cos\phi} + \frac{x^2-a^2}{(x^2+a^2-2ax\cos\phi)^2} \right) d\phi.$$

But
$$\int_0^\pi \frac{d\phi}{x^2+a^2-2ax\cos\phi} = \frac{\pi}{x^2-a^2} \text{ or } \frac{\pi}{a^2-x^2},$$

according as $x >$ or $< a$, and

$$\int_0^\pi \frac{d\phi}{(x^2+a^2-2ax\cos\phi)^2} = \frac{\pi(a^2+x^2)}{(x^2-a^2)^3} \text{ or } \frac{\pi(a^2+x^2)}{(a^2-x^2)^3}.$$

Hence the whole attraction

$$= \pm \frac{\mu a}{x} \left(\frac{\pi}{x^2-a^2} + \frac{\pi(a^2+x^2)}{(x^2-a^2)^2} \right) = \pm \frac{2\mu\pi a x}{(x^2-a^2)^2};$$

hence, taking the upper sign, if the particle be projected in a direction opposite to the resultant force

$$\frac{d^2x}{dt^2} = -\frac{2\mu\pi a x}{(x^2-a^2)^2}, \quad \text{whence} \quad (\text{vel.})^2 = \frac{2\mu\pi a}{x^2-a^2},$$

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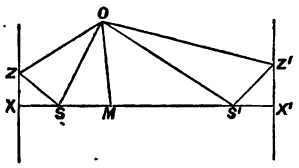
since the velocity has to vanish when $x = \infty$; hence if projected in any other direction the velocity at any distance from the centre of the circle will be the same as before, and the equation of the path described is therefore $\frac{h^2}{p^2} = \frac{2\mu\pi a}{r^2 - a^2}$, which is an epicycloid concentric with the ring and having its cusps on the ring.

3894. (Proposed by C. TAYLOR, M.A.)—Prove that the polar of a point with respect to a conic meets the directrices on the tangents from that point to a confocal.

I. *Solution by* F. D. THOMSON, M.A.

Let O be the given point; S, S′ the foci; Z, Z′ the points where the polar of O meets the directrices; then OSZ, OS′Z′ are right angles.



Draw OM perpendicular to XSS′X′; then $\frac{ZS}{SO} = \frac{ZS}{SX} \cdot \frac{SX}{SO} = \frac{SO}{OM} \cdot \frac{SX}{SO} = \frac{SX}{OM}$ by similar triangles.

Similarly $\frac{Z'S'}{S'O} = \frac{S'X'}{OM} = \frac{SX}{OM}$, $\therefore \frac{ZS}{SO} = \frac{Z'S'}{S'O}$, or $\angle ZOS = \angle Z'OS'$; therefore OZ and OZ′ are tangents to a confocal.

II. *Solution by* Professor TOWNSEND, F.R.S.

Denoting by O the given point, by Y and Y′ the intersections of its polar with the curve, and by Z and Z′ the intersections of the same with the two directrices corresponding respectively to the two foci S and S′; then, since the two angles SOZ and S′OZ′ divide harmonically the angle YOY′, and since angle SOY = angle S′OY′, therefore angle SOZ = angle S′OZ′, and therefore, &c.

3979. (Proposed by J. J. SYLVESTER, F.R.S.)—If from the angles of a tetrahedron lines be drawn parallel to those joining the centroids of the opposite faces with (O) any fixed point in space, prove (1) that they will meet in a point I lying in the line (produced) which passes through the fixed point and the centroid (G) of the tetrahedron, and (2) that in the line OGI so obtained GI = 3OG.

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I. *Solution by* Dr. HOPKINSON; R. F. SCOTT; *and others.*

Let $\alpha, \beta, \gamma, \delta$ be the vectors of the angular points referred to O.

The vector equation of the parallel through the point (α) to the line joining O to the centre of gravity of the face ($\beta\gamma\delta$), *i.e.*, to the vector $\frac{1}{3}(\beta+\gamma+\delta)$, is $\rho = \alpha + \frac{1}{3}(\beta+\gamma+\delta)x$. This passes through a point I whose vector is $\alpha+\beta+\gamma+\delta$; and by symmetry we see that the other parallels do so too. Now OG is $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$, therefore OGI is a straight line, and GI = 3OG.

[The PROPOSER'S proof is as follows:—The resultant of four forces OA, OB, OC, OD will be the resultant of OA, and 3Oa, where a is the centroid of BCD; consequently its extremity will lie in a line through A parallel to Oa; and so for all the other angles. Moreover the resultant of OA, OB, OC, OD is represented in magnitude and direction by 4OG, so that OI = 4OG, and GI = 3OG.

Mr. SCOTT remarks that if, through the centroid of each face, a line be drawn parallel to the line from O to the opposite vertex, these lines will meet in a point given by the vector $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$. If any sixteen points in space be taken and combined in sets of four tetrahedrons, and if for each tetrahedron we find the I point, the I point corresponding to the tetrahedron determined by these four points is independent of the way in which we combine the points, provided all the points be included in each combination.]

II. *Solution by* Professor WOLSTENHOLME.

If A, B, C, D be the corners of the tetrahedron; a, b, c, d the centroids of the opposite faces; Aa, Bb, Cc, Dd meet in G the centre of inertia of both tetrahedrons ABCD, *abcd*; also G is a centre of similitude to the two, any point P belonging to ABCD having a corresponding point p belonging to *abcd*, such that P, G, p are in one straight line, and PG = 3Gp. Hence if AP, BP, CP, DP be drawn, ap, bp, cp, dp will be parallel to them. Therefore, &c.

It will be noticed that the proof only depends on the two tetrahedrons being similar and similarly placed, and that the proposition is equally true if *abcd* be any tetrahedron whose faces are parallel to the corresponding faces of ABCD, except that the ratio will be altered from 3 : 1.

This proposition may be generalized after the method which, in two dimensions, is called Projections; the extended proposition being:—ABCD, *abcd* are two tetrahedrons such that Aa, Bb, Cc, Dd meet in a point G, and therefore the intersections of corresponding faces of the two lie in a plane (G'), then if P be any point, and PA, PB, PC, PD meet the plane G' in points $\alpha, \beta, \gamma, \delta$, then will $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ meet in another point p.

3755. (Proposed by S. WATSON.)—From any point P in a diameter of a given sphere, perpendiculars PQ, PR are drawn of any length and in any direction, so that P and Q may lie within the sphere; and through P, Q, R two spheres are drawn to touch the given one; show that the sum of their radii is equal to the radius of the given sphere.

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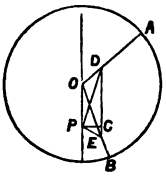
I. *Solution by* ASHER B. EVANS, M.A.

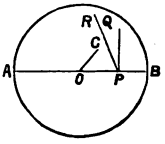


Let C be the centre of the circle passing through P, Q, R; D and E the centres of the two spheres, and A and B their respective points of tangency with the given sphere whose centre is O. Since the locus of all points equally distant from P, Q, R is a line passing through C and parallel to PO, we may suppose the points O, P, E, C, D, A, B to be in the plane of the paper. Since D is in OA, E in OB, C in DE, and PD = AD, and PE = EB, we must have

$$OD + PD = OE + PE = OA.$$

Since the triangles DOP, EOP have the same base and altitude, and the sum of their other two sides equal to the same constant quantity, they are equal in all respects; hence we have OD = PE, OE = PD. But PE is the radius of one of the required spheres, and PD the radius of the other; therefore $OD + PD = PE + PD = OA$.

II. *Solution by the* PROPOSER.



Let O be the centre, and AB a diameter of the given sphere, and C the centre of the sphere through P, Q, R. Take O for origin of rectangular coordinates, OB the axis of xz, and PQ in the plane of xz. Put $OB = a$, $OP = x$, $PQ = z$, $PR = y$, $\angle QPR = \theta$, $OC = m$, $CP = CQ = CR = r$, and u, v, w for the direction cosines of OC. Then the coordinates are

$$P = (x, 0, 0), \quad Q = (x, 0, z),$$
$$R = (x, y \sin\theta, y \cos\theta), \quad C = (mu, mv, mw).$$

Therefore
$$r^2 = (x - mu)^2 + m^2v^2 + (z - mw)^2$$
$$= (x - mu)^2 + (y \sin\theta - mv)^2 + (y \cos\theta - mw)^2.$$

From these, since $u^2 + v^2 + w^2 = 1$, we find successively,

$$2mxu = x^2 + m^2 - r^2 = a^2 + x^2 - 2ar, \text{ since } m = a - r,$$
$$2mxw = xz; \quad 2mx \sin\theta v = x(y - z\cos\theta).$$

Multiply the first two of these by $\sin\theta$, square, and add, the result gives

$$4(a^2 - x^2)\sin^2\theta(r^2 - ar) + (a^2 + x^2)^2 \sin^2\theta + x^2(y^2 + z^2) - 2x^2yz\cos\theta = 0.$$

This quadratic shows that two spheres can be drawn through P, Q, R to touch the given one, and by the theory of equations the sum of their radii is equal to a.

3946. (Proposed by MATTHEW COLLINS, B.A.)—Find (1) the five-point parabolic asymptote of the quartic curve

$$x^2y^2 + axy^2 + bx^2y + cx^3 + d^2xy + e^2x^2 + f^3y = 0,$$

(2) the coordinates of the centre of gravity of the triangle formed by the other three points common to both curves, and (3) the equation of its circumscribed circle.

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Solution by Professor WOLSTENHOLME.

The equation of the quartic may be written

$$x(x+a)(y^2+cx+by+e^2-ac)+(d^2-ab)xy-a(e^2-ac)x+f^2y=0,$$

whence the parabola $y^2+cx+by+e^2-ac=0$ has five-pointic contact with the quartic at infinity, since this parabola and the hyperbola

$$(d^2-ab)xy-a(e^2-ac)x+f^2y=0$$

have one common point at infinity.

[Mr. COLLINS remarks that this quartic is the projection of another whose double point and a tangent drawn from it to the curve are projected to infinity; and that CRAMER does not find the closest parabolic asymptote in page 167 of his work on curves.]

3892. (Proposed by J. J. WALKER, M.A.)—Show that the locus of the intersection of tangents at the extremities of a normal chord of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$.

Solution by T. MITCHESON, B.A.; R. F. SCOTT; *and many others.*

The equation to the normal at the point whose eccentric angle is θ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

If this be the polar of (X, Y), we have

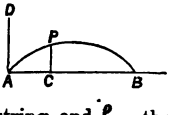
$$\frac{X}{a^2}=\frac{a}{c^2\cos\theta},\text{ or }c^2\cos\theta=\frac{a^3}{X};\quad\frac{Y}{b^2}=-\frac{b}{c^2\sin\theta},\text{ or }c^2\sin\theta=-\frac{b^3}{Y}.$$

Hence the locus required is $\frac{a^6}{X^2}+\frac{b^6}{Y^2}=c^4=(a^2-b^2)^2$.

3896. (Proposed by G. S. CARR.)—A uniform rod of length a, elastic only to flexure, has its ends fastened to a string of length $b<a$; determine the nature of the curve.

Solution by ASHER B. EVANS, M.A.

Let AB represent the string, and APB the curve. Take A as the origin, and AB, AD the axes of rectangular coordinates. Let P be any point in the curve, ρ the radius of curvature at this point, ψ the angle made by the tangent at P with AB, AC = x, CP = y, the arc AP = s; also let t = the tension of the string and $\frac{\rho}{\epsilon}$ = the elasticity of the arc at P.



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When equilibrium exists, the elasticity at any point of the curve must balance the moment of the tension with respect to that point, therefore

$$\frac{\epsilon}{\rho} = ty \quad \ldots\ldots (1),$$

where ϵ and t are constants. Differentiating (1) with respect to s, we find

$$-\frac{\epsilon}{\rho^2}\cdot\frac{d\rho}{ds} = t\cdot\frac{dy}{ds} \quad \ldots\ldots (2).$$

Observing that $\rho = -\dfrac{ds}{d\psi}$ and $\dfrac{dy}{ds} = \sin\psi$,

(2) gives $\dfrac{\epsilon d\rho}{\rho^2} = t\sin\psi\, d\psi$; therefore $\dfrac{\epsilon}{2\rho^2} = t(\cos\psi - \cos\psi_1)$ ..... (3),

where ψ_1 is the value of ψ when $y = 0$, and therefore when [see (1)] $\dfrac{1}{\rho} = 0$. Put $\dfrac{\epsilon}{2t} = m^2$; then, from (3), we find for the equation of the required curve

$$\rho = \frac{m}{(\cos\psi - \cos\psi_1)^{\frac{1}{2}}}.$$

To determine $\cos\psi_1$ we have

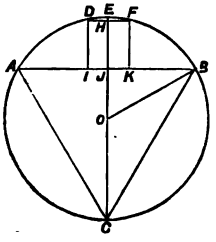
$$\tfrac{1}{2}a = \int ds = -m\int_{\psi_1}^{0} \frac{d\psi}{(\cos\psi - \cos\psi_1)^{\frac{1}{2}}}, \text{ since } ds = \rho d\psi.$$

[A Solution by Dr. BALL is given on p. 66 of Vol. XVIII. of the *Reprint*.]

3850. (Proposed by the EDITOR.)—Prove that the side of a square inscribed in one of the segments of a circle cut off by the side of an inscribed equilateral triangle of side a is $\frac{1}{15}(\sqrt{57} - 2\sqrt{3})\,a$, or $(\cdot 27238)\,a$.

Solution by A. MARTIN; W. PRIESTLEY; *and many others.*

Let $AB = a$ = side of the triangle, and $DF = x$ = side of the square. Then, in the figure, we have $CJ = \frac{1}{2}a\sqrt{3}$, $OC = \frac{2}{3}CJ = \frac{1}{3}a\sqrt{3}$,
$OJ = EJ = \frac{1}{3}CJ = \frac{1}{6}a\sqrt{3}$, $HE = \frac{1}{6}a\sqrt{3} - x$,
$HC = \frac{1}{6}a\sqrt{3} + x$, $DH = HF = \frac{1}{2}x$.



Therefore $(\frac{1}{2}x)^2 = (\frac{1}{6}a\sqrt{3} - x)(\frac{1}{6}a\sqrt{3} + x)$;
whence $15x^2 + 4ax\sqrt{3} = 3a^2$,
and $x = \frac{1}{15}(\sqrt{57} - 2\sqrt{3})\,a = (\cdot 27238)\,a$.

3925. (Proposed by G. O'HANLON.)—P is any point in the base BC of the right-angled triangle ABC, and BD, PF are drawn perpendicular to the hypotenuse AC: prove that $AF \cdot AD + BD \cdot PF = AB^2$.

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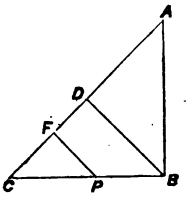
Solution by LIZZIE A. KITTUDGE; E. C. CARNT; *and many others.*



From the similar triangles CPF, BAD we have $CF \cdot AD = BD \cdot PF$; therefore

$$AB^2 = AC \cdot AD = AF \cdot AD + CF \cdot AD$$
$$= AF \cdot AD + BD \cdot PF.$$

3947. (Proposed by S. WATSON.)—If O_1, O_2, O_3 be the escribed centres of the triangle ABC, and ρ_1, ρ_2, ρ_3 the radii of circles inscribed in the triangles BO_1C, CO_2A, AO_3B; prove that

$$\left(\frac{a}{\rho_1} + \frac{b}{\rho_2} + \frac{c}{\rho_3}\right)^{\frac{1}{2}} = \left(\frac{a}{r_1}\right)^{\frac{1}{2}} + \left(\frac{b}{r_2}\right)^{\frac{1}{2}} + \left(\frac{c}{r_3}\right)^{\frac{1}{2}}.$$

Solution by A. ESCOTT, M.A.; A. B. EVANS, M.A.; T. MITCHESON, B.A. *and many others.*

Since $BO_1 = \frac{r_1}{\cos \frac{1}{2}B}$, $CO_1 = \frac{r_1}{\cos \frac{1}{2}C}$, we have

$$2\Delta BO_1C = ar_1 = (BC + BO_1 + CO_1)\,\rho_1 = \left(a + \frac{r_1}{\cos \frac{1}{2}B} + \frac{r_1}{\cos \frac{1}{2}B}\right)\rho_1;$$

whence $$\frac{a}{\rho_1} = \frac{a}{r_1} + \frac{1}{\cos \frac{1}{2}B} + \frac{1}{\cos \frac{1}{2}C}, \text{ \&c., \&c.;}$$

therefore $$\frac{a}{\rho_1} + \frac{b}{\rho_2} + \frac{c}{\rho_3} = \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} + \frac{2}{\cos \frac{1}{2}A} + \frac{2}{\cos \frac{1}{2}B} + \frac{2}{\cos \frac{1}{2}C},$$

which readily reduces to the required form, since

$$\frac{a}{r_1} = \frac{\cos \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C}, \quad \frac{b}{r_2} = \frac{\cos \frac{1}{2}B}{\cos \frac{1}{2}A \cos \frac{1}{2}C}, \quad \frac{c}{r_3} = \frac{\cos \frac{1}{2}C}{\cos \frac{1}{2}A \cos \frac{1}{2}B},$$

therefore $$\left(\frac{ab}{r_1 r_2}\right)^{\frac{1}{2}} = \frac{1}{\cos \frac{1}{2}C}, \quad \left(\frac{ac}{r_1 r_3}\right)^{\frac{1}{2}} = \frac{1}{\cos \frac{1}{2}B}, \quad \left(\frac{bc}{r_2 r_3}\right)^{\frac{1}{2}} = \frac{1}{\cos \frac{1}{2}A}.$$

3901. (Proposed by S. BILLS.)—Given the base, the vertical angle, and the distance from the centre of the circumscribed circle to the centroid of a plane triangle; to construct the triangle.

I. *Solution by* H. MURPHY; A. B. EVANS, M.A.; *the* PROPOSER; *and many others.*

From the centre of the circumscribed circle inflect a line, equal to the given distance, upon the segment on the middle third part of the base containing an angle equal to the vertical angle, and we obtain the centroid.

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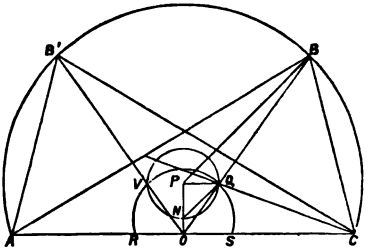
II. *Solution by the* EDITOR.



On the given base AC construct a circular segment AB'BC containing an angle equal to the given vertical angle; and from P, the centre of this circle, draw PO perpendicular to AC. Take ON = $\frac{1}{3}$ OP, and from N as centre, with $\frac{1}{3}$ PA as radius, draw the circular segment RVQS, which is clearly *similar* to AB'BC and one-third of it in *linear* dimensions; also from P as centre, with the given distance of the centroid from P as radius, draw the circle QV, cutting RVQS in Q and V. Join OQ, OV, and produce these lines to meet the arc AB'BC in B and B'; also join AB, BC, AB', B'C.

Then each of the triangles ABC, AB'C obviously satisfies the conditions of the problem.

3727. (Proposed by G. S. CARR.)—Show that the shape of a uniform plane elastic lamina, which will take a circular form when its ends are drawn together to meet, may be obtained by drawing two cutting planes through a tangent which is at right angles to the axis of a cylinder, and unrolling the intercepted part of the surface.

Solution by the PROPOSER.

Let the circle APB represent the bent lamina, its ends meeting at A. Let the ends be kept together by a tension T acting in the tangent at A. Let P be any point on the circle, and PN perpendicular to the tangent AN.

Then the moment of the strain at P = T . PN; and the moment of the interior forces in equilibrium with this at P varies as the width of the lamina ÷ the radius of curvature at P; and this radius is constant; therefore the width $\propto$ PN.

But two planes passing through AN will intercept a portion of the surface of the cylinder whose section is APB, so that the width of the portion $\propto$ PN, and will therefore give the shape required.

[A Solution by Professor TOWNSEND is given on p. 108 of Vol. XVII. of the *Reprint*.]

2108. (Proposed by S. BILLS.)—One sovereign, two half-sovereigns, one crown, two half-crowns, one shilling, two sixpences, one penny, and two half-pennies, are put into a bag, from which three persons, A, B, C, draw out, in succession, each of them a coin. Show that the value of the expectation of each is 4*s*. 4$\frac{1}{3}$*d*.

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Solution by the PROPOSER; F. SINCLAIR; *and others.*

More generally; let there be n coins, and let their values be

$$a_1, a_2, \ldots\ldots, a_n,$$ and their total value $= s$, suppose.

Then A's expectation is evidently $= \frac{s}{n}$;

B's will be $$= \frac{1}{n}\left(\frac{s-a_1}{n-1} + \frac{s-a_2}{n-1} + \ldots\ldots + \frac{s-a_n}{n-1}\right) = \frac{s}{n};$$

C's will be $$= \frac{2}{n(n-1)}\left(\frac{s-a_1-a_2}{n-2} + \frac{s-a_1-a_3}{n-2} + \&c.\right)$$

$$= \frac{n(n-1)s - 2(n-1)s}{n(n-1)(n-2)} = \frac{s}{n}.$$

Therefore, the *à priori* expectation of each is the same and $= \frac{s}{n}$.

In the particular case proposed in the Question, we have $n = 12$, $s = 2l.\ 12s.\ 2d.$ Therefore the expectation of each is $4s.\ 4\frac{1}{6}d.$

3815. (Proposed by J. J. WALKER, M.A.)—Show that the locus of feet of perpendiculars let fall from points in a given diameter of a quadric surface on the polar planes of those points is a rectangular hyperbola.

Solution by R. W. GENESE, B.A.

The polar planes are parallel to one another, therefore the perpendiculars lie in a plane through the given diameter perpendicular to them. Let this plane meet the quadric in the conic S, the polar plane of P in QY, and PY be perpendicular from P on QY. Then QY is the polar of P with respect to S. Let CX be the perpendicular from the centre C of the conic on QY. Then CX : CQ and XY : CP are constant ratios. But CP . CQ is constant, therefore CX . XY is constant, and the locus of Y is a rectangular hyperbola whose asymptotes are CX and a perpendicular through C.

[An algebraical proof is given on p. 41 of Vol. XVIII. of the *Reprint*.]

2213. (Proposed by the EDITOR.)—A conic passes through the angular points and the centroid of a given triangle; prove that the locus of its centre is the maximum ellipse that can be inscribed in the triangle, which touches the sides at their middle points, and has an area of $\frac{1}{3}\pi\sqrt{3} \cdot \Delta$.

I. *Solution by the* PROPOSER.

Using triangular coordinates (x, y, z) let the equation of the conic through the angular points be

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

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then, since this passes through the centroid, where $x=y=z=\frac{1}{3}$, we have

$$l+m+n=0 \quad \ldots\ldots (2).$$

Now the centre (x', y', z') of (1) is given by (See WHITWORTH's *Modern Analytical Geometry*, Art. 194)

$$\frac{x'}{l^2}=\frac{y'}{m^2}=\frac{z'}{n^2} \quad \ldots\ldots (3).$$

Hence, substituting from (3) in (2), and suppressing the accents, the locus of the centre of (1) is given by

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0 \quad \ldots\ldots (4),$$

which designates the ellipse specified in the question. (See the Solution of Quest. 2364, on p. 29 of Vol. X. of the *Reprint*.)

II. *Solution by* STEPHEN WATSON.

Take BC$=a$, BA$=c$, two of the sides of the triangle, for axes; then the coordinates of the centroid are $\frac{1}{3}a$, $\frac{1}{3}c$, and the equation of a conic through A, B, C is

$$x^2+mxy+cny^2-ax-cny=0 \quad \ldots\ldots (1);$$

hence, when this passes through the centroid, we have

$$acm-2c^2n=2a^2 \quad \ldots\ldots (2).$$

Now the coordinates x, y of the centre of (1) are

$$x=\frac{n(cn-2a)}{m^2-4n}=\frac{2c^2n^2}{a(m^2-4n)} \text{ by (2)} \quad \ldots\ldots (3),$$

$$y=\frac{ma-2cn}{m^2-4n}=\frac{2a^2}{c(m^2-4n)} \quad \ldots\ldots (4);$$

hence, eliminating m and n from (2), (3), (4), we get

$$(ac-2ay-2cx)^2=4acxy,$$

which is the equation of an ellipse touching the triangle at the point of bisection of each side; it is therefore the *maximum* inscribed ellipse, and its area is $\frac{\pi}{3^{\frac{1}{2}}}$ of the triangle.

3876. (Proposed by J. J. SYLVESTER, F.R.S.)—Show that there are 3184 positions in a cubic curve such that at each of them curves of the 50th order may be drawn having 90-point contact with the cubic.

Solution by Professor CLIFFORD.

The gross number of points where a curve of order n can have $3n$-point contact with a cubic is $9n^2$. The problem is in fact the same as that of the division of the periods of an elliptic function by $3n$, and as there are two periods, there are $9n^2$ solutions. (CLEBSCH, *Anwendung der Abelschen Functionen in der Geometrie*, Crelle, lxiii.) But in the case when n is a composite number, all the curves whose order is a divisor of n, and which have complete contact with the cubic, are included in the result. Thus

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each inflexional tangent, taken n times over, constitutes a curve of order n having $3n$-point contact with the cubic; and the nine inflexional tangents are thus always included in the $9n^2$ solutions. To obtain the number of *proper* solutions, then, we must subtract all these improper ones. When $n=30$, the result is

$$9\left\{30^2-15^2-10^2-6^2+5^2+3^2+2^2-1^2\right\} = 9\times 576 = 5184.$$

Here it is to be observed that the curves of order 5 are *twice* subtracted, with the curves of order 15 and 10; so that they have to be added in again. The same remark applies to the orders 3 and 2. The curves of order 1 (inflexional tangents) having been thrice subtracted and thrice added, must finally be subtracted again.

2011. (Proposed by the EDITOR.)—Show that the trilinear equation of the circle which passes through the points where the internal bisectors of the angles of the triangle of reference meet the sides, is

$$(u+\cos A)\,\alpha^2+(v+\cos B)\,\beta^2+(w+\cos C)\,\gamma^2 = (v+w+\cos B+\cos C)\,\beta\gamma + (u+w+\cos A+\cos C)\,\gamma\alpha+(u+v+\cos A+\cos B)\,\alpha\beta,$$

where $u=\frac{a}{2s}$, $v=\frac{b}{2s}$, $w=\frac{c}{2s}$, and $2s=a+b+c$.

Solution by STEPHEN WATSON.

The trilinear equations of the lines joining the three points where the internal bisectors meet the opposite sides, are

$$-\alpha+\beta+\gamma=0,\quad \alpha-\beta+\gamma=0,\quad \alpha+\beta-\gamma=0;$$

hence the equation of a circle through these points may be written

$$l\left\{\alpha^2-(\beta-\gamma)^2\right\}+m\left\{\beta^2-(\gamma-\alpha)^2\right\}+n\left\{\gamma^2-(\alpha-\beta)^2\right\}=0 \ldots\ldots (1),$$

subject to the condition that when $\alpha = x\cos\alpha+y\sin\alpha-p$, with similar values for β and γ, the coefficients of x^2 and y^2 shall be equal, and that of xy shall be zero. These conditions give

$$\begin{aligned} &l\left\{\cos 2\alpha-\cos 2\beta-\cos 2\gamma+2\cos(\beta+\gamma)\right\}\\ &+m\left\{\cos 2\beta-\cos 2\gamma-\cos 2\alpha+2\cos(\gamma+\alpha)\right\}\\ &+n\left\{\cos 2\gamma-\cos 2\alpha-\cos 2\beta+2\cos(\alpha+\beta)\right\}=0,\\ &l\left\{\sin 2\alpha-\sin 2\beta-\sin 2\gamma+2\sin(\beta+\gamma)\right\}\\ &+m\left\{\sin 2\beta-\sin 2\gamma-\sin 2\alpha+2\sin(\gamma+\alpha)\right\}\\ &+n\left\{\sin 2\gamma-\sin 2\alpha-\sin 2\beta+2\sin(\alpha+\beta)\right\}=0. \end{aligned}$$

Eliminating n and m separately from these equations, and reducing,

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we find

$$\frac{l}{v+w+\cos B+\cos C} = \frac{m}{w+u+\cos C+\cos A} = \frac{n}{u+v+\cos A+\cos B};$$

and substituting these values in (1), we obtain the equation in the form given in the question.

[Putting $\alpha=0$ in the equation of the circle, we obtain

$$(\beta-\gamma)\left\{(v+\cos B)\beta-(w+\cos C)\gamma\right\} = 0,$$

which shows that the circle cuts BC in the point where it is cut by the internal bisector ($\beta-\gamma=0$) of the angle A, and also in another point such that the perpendiculars from it on AB, AC are as

$$v+\cos B : w+\cos C, \text{ or as } b+2s\cos B : c+2s\cos C.]$$

2105. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Prove that

$$i\sin\left\{i\log_e\left[x+(x^2+1)^{\frac{1}{2}}\right]\right\}\cos\left\{i\log_e\left[x+(x^2-1)^{\frac{1}{2}}\right]\right\} = -x^2,$$

where i is the square root of -1.

Solution by the Proposer; S. Watson; S. Bills; *and others.*

The well known formulæ

$$i\sin\theta = \tfrac{1}{2}(e^{i\theta}-e^{-i\theta}), \quad \cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}),$$

are equivalent to the following:

$$i\sin(i\theta) = \tfrac{1}{2}(e^{-\theta}-e^{\theta}), \quad \cos(i\theta) = \tfrac{1}{2}(e^{-\theta}+e^{\theta}).$$

Let $e^{\alpha} = x+(x^2+1)^{\frac{1}{2}}$, then $e^{-\alpha} = -x+(x^2+1)^{\frac{1}{2}}$;

hence the first factor of the proposed expression becomes

$$i\sin(i\alpha) = \tfrac{1}{2}(e^{-\alpha}-e^{\alpha}) = -x.$$

Again, let $e^{\beta} = x+(x^2-1)^{\frac{1}{2}}$, then $e^{-\beta} = x-(x^2-1)^{\frac{1}{2}}$;

hence, by substitution, the second factor becomes

$$\cos(i\beta) = \tfrac{1}{2}(e^{-\beta}+e^{\beta}) = x.$$

Therefore the proposed expression reduces simply to $-x\times x = -x^2$.

1230. (Proposed by the Editor.)—In a given triangle ABC another triangle DEF is inscribed, by taking a random point in each of the sides; show that the probability that the triangle DEF does not exceed half the triangle ABC is $\frac{1}{12}(6-\pi)+\log 2$, or ·9313478, or $\frac{14}{15}$ nearly.

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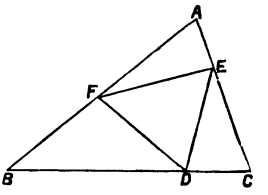
Solution by the PROPOSER.

Put BD$=x$, CE$=y$, BF$=z$; then when $\triangle DEF = \frac{1}{2}\triangle ABC$, we have

$$\triangle BDF + \triangle CDE + \triangle AEF$$
$$= \frac{\sin B}{2}\left\{xz + \frac{cy(a-x)}{b} + \frac{a(b-y)(c-z)}{b}\right\}$$
$$= \tfrac{1}{4}ac\sin B;$$



therefore $y = \mathrm{CE} = \dfrac{\left\{(a-x)z - \frac{1}{2}ac\right\}b}{az-cx}$,

and $$\mathrm{AE} = \frac{\left\{(c-z)x - \frac{1}{2}ac\right\}b}{cx-az};$$

hence, in order that CE may exceed 0, we must have $z > \dfrac{ac}{2(a-x)}$, and therefore $x < \frac{1}{2}a$; also, in order that AE may exceed 0, we must have $x > \dfrac{ac}{2(c-z)}$, and therefore $z < \frac{1}{2}c$. Consequently the chance (q) that $\triangle DEF$ shall be $> \frac{1}{2}\triangle ABC$, is

$$q = \frac{1}{abc}\left\{\int_0^{\frac{1}{2}a}\int_{\frac{ac}{2(a-x)}}^{c}(\mathrm{CE})\,dx\,dz + \int_0^{\frac{1}{2}c}\int_{\frac{ac}{2(a-z)}}^{a}(\mathrm{AE})\,dz\,dx\right\}.$$

Now
$$\frac{1}{abc}\int_0^{\frac{1}{2}a}\int_{\frac{ac}{2(a-x)}}^{c}(\mathrm{CE})\,dx\,dz$$
$$= \frac{1}{2a^2}\int_0^{\frac{1}{2}a}\left\{a - 2x + \frac{(a^2-2ax+2x^2)}{a}\log\frac{a^2-2ax+2x^2}{2(a-x)^2}\right\}dx$$
$$= \frac{1}{8} - \frac{1}{2a^3}\int_0^{\frac{1}{2}a}(a^2x - ax^2 + \tfrac{2}{3}x^3)\,d\log\frac{a^2-2ax+2x^2}{2(a-x)^2}$$
$$= \frac{1}{8} - \frac{1}{a^2}\int_0^{\frac{1}{2}a}\frac{(a^2-ax+\frac{2}{3}x^2)x^2\,dx}{(a-x)(a^2-2ax+2x^2)}$$
$$= \frac{1}{8} + \frac{1}{a^2}\int_0^{\frac{1}{2}a}\left\{\tfrac{1}{6}a + \tfrac{1}{3}x - \frac{2a^2}{3}\cdot\frac{1}{a-x} + \frac{a^2(\frac{1}{2}a - \frac{2}{3}x)}{a^2-2ax+2x^2}\right\}dx$$
$$= \tfrac{1}{4} - \tfrac{1}{2}\log 2 + \tfrac{1}{24}\pi.$$

In like manner we have

$$\frac{1}{abc}\int_0^{\frac{1}{2}c}\int_{\frac{ac}{2(a-z)}}^{a}(\mathrm{AE})\,dz\,dx = \tfrac{1}{4} - \tfrac{1}{2}\log 2 + \tfrac{1}{24}\pi;$$

therefore $q = \frac{1}{12}(6+\pi) - \log 2$, and $p = \frac{1}{12}(6-\pi) + \log 2$.

[The EDITOR's Solution of this problem is given on p. 96 of Vol. II. of the *Reprint;* and Solutions of a more general problem (Quest. 2612), including this as a particular case, are given on pp. 109—111 of Vol. X. of the *Reprint.*]

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3921. (Proposed by P. O'CAVANAGH.)—Find the six intersections real or imaginary of Descartes' folium $x^3 - 3axy + y^3 = 0$ with the ellipse $x^2 - xy + y^2 - a(x+y) + a^2 = 0$.

Solution by Professor WOLSTENHOLME.

Proceeding to eliminate a, we obtain, successively,

$$(x^3+y^3)(x+y) = 3axy(x+y) = 3xy(x^2-xy+y^2+a^2),$$
$$x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 = 3a^2xy,$$
$$(x^3+y^3)^2 = 9a^2x^2y^2 = 3xy(x^2-xy+y^2)^2,$$
$$(x+y)^2(x^2-xy+y^2)^2 = 3xy(x^2-xy+y^2)^2, \quad (x^2-xy+y^2)^3 = 0;$$

hence the curves osculate at the two impossible points $x+y=a$, $3xy=a^2$.

3852. (Proposed by Professor TOWNSEND, F.R.S.)—Show that a variable tangent to the curve of intersection of any two fixed quadrics determines with the four vertices of their common self-reciprocal tetrahedron a pencil of four planes having a constant anharmonic ratio.

Solution by the PROPOSER.

If $U=0$, $V=0$, be the equations of the two quadrics, and $P=0$, $Q=0$, those of the two polar planes of any point O with respect to them, then since, for any four quadrics, $U+k_1V=0$, $U+k_2V=0$, $U+k_3V=0$, $U+k_4V=0$, passing through their curve of intersection, the four polar planes of O, viz., $P+k_1Q=0$, $P+k_2Q=0$, $P+k_3Q=0$, $P+k_4Q=0$, have an harmonic ratio depending only on the values of the four parameters k_1, k_2, k_3, k_4, and independent consequently of the position of O, that ratio is therefore constant when k_1, &c., are the four roots of the discriminant of $U+kV=0$, and when O is a variable point on the curve of intersection of U and V; but in that case the four polar planes, being the four tangent planes at the point to the four cones which pass through the curve of intersection, are consequently the four planes in question, and therefore, &c.

3843. (Proposed by M. COLLINS, B.A.)—If a whole number m divides a^n-1; prove that m^2 divides $a^{mn}-1$, that m^3 divides $a^{m^2n}-1$, and that m^{r+1} divides $a^{m^r n}-1$.

Solution by ASHER B. EVANS, M.A.

Let $a^n-1 = mA$; then $a^n = 1+mA$, and

$$a^{m^r n} = (a^n)^{m^r} = (1+mA)^{m^r} = 1 + m^r(mA) + \text{terms containing higher powers of } m;$$

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let $m^{r+2}B$ represent these omitted terms, where B is evidently an integer; then

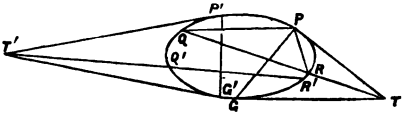
$$a^{m^r n} - 1 = m^{r+1}(A + mB).$$

Taking $r = 1, 2$, we have

$$a^{mn} - 1 = m^2(A + mB), \quad a^{m^2 n} - 1 = m^3(A + mB).$$

3945. (Proposed by F. D. THOMSON, M.A.)—P, P′ are fixed points on one of a series of confocal conics, and from P, P′ tangents are drawn to another conic of the series, meeting the first conic in Q, R; Q′, R′. Show that the locus of the intersection of QR, Q′R′ is a conic.

Solution by the PROPOSER.



Draw the normals PG, P′G′. Then since PQ, PR make equal angles with the normal, QR is cut harmonically by the normal and tangent at P. Hence QR passes through the pole of the normal, or QR passes through a *fixed point* T.

Similarly it may be shown that Q′R′ passes through a fixed point T′.

Also to any one line TRQ of the first series corresponds one line, and one only, of the second series. Hence the two series form homographic pencils, and their intersection lies on a conic through TT′.

The theorem may of course be extended to a series of conics inscribed in the same quadrilateral.

If these conics be referred to their self-conjugate triangle, and if

$$Ax^2 + By^2 + Cz^2 = 0$$

be the equation to the first conic, it will be found that the locus of T, when P moves on the first conic, is

$$\frac{(B-C)^2}{Ax^2} + \frac{(C-A)^2}{By^2} + \frac{(A-B)^2}{Cz^2} = 0,$$

and that the locus in the question is the conic

$$\frac{B-C}{Ax}(y'z'' - z'y'') + \&c. = 0,$$

where (x', y', z'), (x'', y'', z'') are the points P and P′.

[Professor WOLSTENHOLME remarks that, if the equation of the original ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and if α, α' be the excentric angles of P, P′, the equation of the locus is very easily found to be

$$\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1\right)\left(\frac{x}{a}\cos\alpha' - \frac{y}{b}\sin\alpha' - \frac{a^2+b^2}{a^2-b^2}\right)$$
$$= \left(\frac{x}{a}\cos\alpha' + \frac{y}{b}\sin\alpha' - 1\right)\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha - \frac{a^2+b^2}{a^2-b^2}\right),$$

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a rectangular hyperbola having its asymptotes parallel to the axis, its centre at the point $\frac{a^3}{a^2-b^2}\frac{\cos\frac{1}{2}(\alpha+\alpha')}{\cos\frac{1}{2}(\alpha-\alpha')}$, $\frac{b^3}{b^2-a^2}\frac{\sin\frac{1}{2}(\alpha+\alpha')}{\cos\frac{1}{2}(\alpha-\alpha')}$, and passing through the origin; hence completely determined.]

4007. (Proposed by the EDITOR.)—Show that the equation whose roots are the differences of the roots of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

is $x^{12}-48\,\mathrm{H}x^{10}+8\,(96\,\mathrm{H}^2+\mathrm{I})\,x^8-32\,(128\,\mathrm{H}^3+16\,\mathrm{HI}+13\,\mathrm{J})\,x^6+16\,(384\,\mathrm{H}^2\mathrm{I}+288\,\mathrm{HJ}-7\,\mathrm{I}^2)\,x^4-1152\,(2\,\mathrm{HI}^2+3\,\mathrm{IJ})\,x^2+256\,(\mathrm{I}^3-27\,\mathrm{J}^2)=0$,

where $\mathrm{H}=b^2-ac$, $\mathrm{I}=ae-4bd+3c^2$, $\mathrm{J}=ace+2bcd-ad^2-eb^2-c^3$ ...(A.)

Solution by SAMUEL BILLS.

Let λ, μ, ν be the roots of the equation $x^3-\mathrm{I}x+2\mathrm{J}=0$ ..... (2); then, if $\alpha, \beta, \gamma, \delta$ denote the roots of (1), we shall have (see Dr. BALL's Paper on Biquadratics in No. 28 of the *Quarterly Journal of Mathematics*),

$$\left.\begin{aligned}
\alpha &= -b+\frac{1}{\sqrt{2}}\left\{-(\lambda+2\mathrm{H})^{\frac{1}{2}}-(\mu+2\mathrm{H})^{\frac{1}{2}}-(\nu+2\mathrm{H})^{\frac{1}{2}}\right\}\\
\beta &= -b+\frac{1}{\sqrt{2}}\left\{-(\lambda+2\mathrm{H})^{\frac{1}{2}}+(\mu+2\mathrm{H})^{\frac{1}{2}}+(\nu+2\mathrm{H})^{\frac{1}{2}}\right\}\\
\gamma &= -b+\frac{1}{\sqrt{2}}\left\{+(\lambda+2\mathrm{H})^{\frac{1}{2}}-(\mu+2\mathrm{H})^{\frac{1}{2}}+(\nu+2\mathrm{H})^{\frac{1}{2}}\right\}\\
\delta &= -b+\frac{1}{\sqrt{2}}\left\{+(\lambda+2\mathrm{H})^{\frac{1}{2}}+(\mu+2\mathrm{H})^{\frac{1}{2}}-(\nu+2\mathrm{H})^{\frac{1}{2}}\right\}
\end{aligned}\right\} \ldots\ldots (\mathrm{B}).$$

Now the required equation of the differences $\alpha-\beta$, $\alpha-\gamma$, &c., will evidently be of the 12th degree, and the roots will be of the form $+n$, $-n$. Put $2\lambda+4\mathrm{H}=p$, $2\mu+4\mathrm{H}=q$, $2\nu+4\mathrm{H}=r$; then, taking the differences of the equations (B), it will be readily seen that the required equation will be the product of the following twelve factors,

$$\left.\begin{aligned}
&x+(p^{\frac{1}{2}}+q^{\frac{1}{2}})\\
&x-(p^{\frac{1}{2}}+q^{\frac{1}{2}})\\
&x+(p^{\frac{1}{2}}-q^{\frac{1}{2}})\\
&x-(p^{\frac{1}{2}}-q^{\frac{1}{2}})
\end{aligned}\right\}\left\{\begin{aligned}
&x+(p^{\frac{1}{2}}+r^{\frac{1}{2}})\\
&x-(p^{\frac{1}{2}}+r^{\frac{1}{2}})\\
&x+(p^{\frac{1}{2}}-r^{\frac{1}{2}})\\
&x-(p^{\frac{1}{2}}-r^{\frac{1}{2}})
\end{aligned}\right\}\left\{\begin{aligned}
&x+(q^{\frac{1}{2}}+r^{\frac{1}{2}})\\
&x-(q^{\frac{1}{2}}+r^{\frac{1}{2}})\\
&x+(q^{\frac{1}{2}}-r^{\frac{1}{2}})\\
&x-(q^{\frac{1}{2}}-r^{\frac{1}{2}})
\end{aligned}\right\} \ldots (\mathrm{C}).$$

Multiplying together the first group of (C), we obtain

$$x^4-2\,(p+q)\,x^2+(p-q)^2=0,$$

or
$$x^4-4\,(\lambda+\mu+4\mathrm{H})\,x^2+4\,(\lambda-\mu)^2=0,$$

which reduces to $x^4+4\,(\nu-4\mathrm{H})\,x^2+4\left(\nu^2+\frac{8\mathrm{J}}{\nu}\right)=0$ ..... (3).

In like manner, from the second and third groups we obtain

$$x^4+4\,(\mu-4\mathrm{H})\,x^2+4\left(\mu^2+\frac{8\mathrm{J}}{\mu}\right)=0 \ldots\ldots (4),$$

$$x^4+4\,(\lambda-4\mathrm{H})\,x^2+4\left(\lambda^2+\frac{8\mathrm{J}}{\lambda}\right)=0 \ldots\ldots (5).$$

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Multiplying together (3), (4), (5), it will be found that the coefficients of the several powers of w will be symmetric functions of λ, μ, ν; and, substituting for those functions their values in terms of I and J, as derived from (2), we obtain, as the required equation, the one given in the question.

[Professor CAYLEY remarks that this theorem is given in a somewhat more general form in his paper "On the Equation of differences for an Equation of any order," in the volume of the *Philosophical Transactions* for 1860, viz., if $U = (a, b, c, d, e)(x, y)^4 = a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)$, and if H is the Hessian, I, J the invariants, then the equation for the determination of $\theta, = \frac{(\alpha-\beta)^2(\alpha-\gamma)^2\ldots}{(x-\alpha y)^2(x-\beta y)^2(x-\gamma y)^2(x-\delta y)^2}$ is

$$\left\{\begin{array}{l} U^6 \\ +48U^4H \\ +8U^2(U^2I+96H^2) \\ +32(-13U^3J+16U^2HI+128H^3) \\ +16(-7U^2I^2-288UHJ+384H^2I) \\ +1152(-3UIJ+2HI^2) \\ +256(I^3-27J^2) \end{array}\right\}(\theta, 1)^6 = 0.$$

Putting herein $U=1$, and changing the sign of H (it would have been better in the question to have written $H=ac-b^2$, instead of b^2-ac), this agrees with the result given in the Question.]

4003. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Two planes are drawn parallel to the circular sections of an oblate spheroid at the distance be^{-1} from the centre; and a secant plane is drawn, cutting the surface in a conic section, and the two parallel planes in two straight lines. Show that the diametral planes through these straight lines are parallel to the circular sections of the cone, whose vertex is the centre and base the conic section in which the surface is cut by the secant plane.

I. *Solution by* R. F. SCOTT.

Let the equations of the spheroid and the secant plane be respectively

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1, \quad lx+my+nz = p;$$

then the equation of the cone in question is

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = \left(\frac{lx+my+nz}{p}\right)^2;$$

and subtracting $\frac{x^2+y^2+z^2}{a^2} = 0$ from this, we get

$$\frac{e^2z^2}{b^2} = \left(\frac{lx+my+nz}{p^2}\right)^2, \text{ or } \frac{lx+my+nz}{p} = \pm\frac{ez}{b},$$

as the equations to the cyclic planes. But these pass through the intersection of the two parallel planes with the secant plane.

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II. *Solution by* Professor TOWNSEND, F.R.S.

By reciprocation from the centre of the oblate spheroid, this property becomes immediately transformed into the well known one, that the connectors of any arbitrary point with the foci of a prolate spheroid are the focal lines of the cone which, from the point as vertex, envelopes the spheroid, and therefore, &c.

3966. (Proposed by Dr. CASEY.)—Prove that (1) if any two circles be inverted from an arbitrary point, the ratio of the square of their common tangent to the rectangle of their diameters is the same for the converse as for the original circles; (2) employ the preceding theorem in proving Terquem's theorem on the nine-point circle, giving a proof that will apply equally whether the triangle be formed by right lines or by arcs of circles; and (3) prove Terquem's Theorem for spherical triangles formed by arcs of great or small circles.

Solution by Professor TOWNSEND, F.R.S.

The first part of this useful property is proved in my *Modern Geometry*, Vol. ii., Art. 408, for the case of intersecting circles, from the consideration that the two ratios in question for the two common tangents are (disregarding signs) equal to the squares of the sine and of the cosine of the semi-angle of intersection of the circles, which angle it is well known undergoes no change in inversion. And it had been previously proved, and was subsequently published in the *Quarterly Journal of Mathematics*, Vol. vii., p. 378, by Dr. CASEY, for the general case of any two circles, from the consideration that the two ratios in question for the two common tangents are two of the three anharmonic ratios of the four points of intersection of the two circles with the line passing through their centres, which anharmonic ratios are easily seen to undergo no change in inversion. As regards the second and third parts, several demonstrations of Terquem's (or rather of Feuerbach's) original theorem, with numerous and very considerable extensions of it, not only in plane and spherical, but also in geometry of three dimensions, have from time to time been given by the same distinguished geometer, some depending on the process of inversion, and some on other considerations. See *Quarterly Journal of Mathematics*, Vol. iv., p. 245, and Vol. v., p. 318; *Proceedings of the Royal Irish Academy*, Vol. ix., p. 396; *Transactions of the Royal Irish Academy*, Vol. 24, p. 1; and *Philosophical Transactions of the Royal Society*, Vol. 161, p. 585.

[Dr. CASEY remarks that he believes the proof of Terquem's theorem, and its extension contained in the foregoing Question, to be more simple and direct than any yet given.]

4018. (Proposed by B. WILLIAMSON, M.A.)—If the equation of a curve be of the form $r^2 - a^2 = mp^2$; prove that the equation of its evolute is

$$r^2 - (1-m)\,a^2 = mp^2.$$

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Solution by the Rev. G. H. HOPKINS, M.A.

Using the polar formulæ for determining the evolute of a curve, and putting (r', p') for the point on the evolute corresponding to (r, p) on the primitive curve, we have

$$\rho = r\frac{dr}{dp} = mp, \quad p'^2 = r^2 - p^2 = a^2 + (m-1)p^2,$$

$$r'^2 = \rho^2 + r^2 - 2\rho p = (m^2 - 2m)p^2 + r^2 = a^2 + m(m-1)p^2,$$

therefore $$mp'^2 - r'^2 = (m-1)a^2,$$

which, dropping the accents, is the equation required.

[If m be less than unity, the curve and the evolute are similar. This establishes a well known property of the Epicycloid.]

3989. (Proposed by G. O'HANLON.)—If a quadrilateral has the sum of the squares of the diagonals equal to the sum of the squares of the sides, prove that it is a plane figure and a parallelogram.

I. *Solution by* CHRISTINE LADD; G. M. MINCHIN, M.A.; *and others.*

Let the vector sides of the quadrilateral be $\alpha, \beta, \gamma, \delta$ in order.

Then, by supposition, we have

$$(\alpha+\beta)^2 + (\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$

But $(\alpha+\beta)^2 = \alpha^2 + 2S\alpha\beta + \beta^2$ and $(\beta+\gamma)^2 = \beta^2 + 2S\beta\gamma + \gamma^2$;

therefore $$\delta^2 = \beta^2 + 2S\alpha\beta + 2S\alpha\gamma.$$

But $\alpha+\beta+\gamma+\delta = 0$; therefore, substituting for δ its value $-(\alpha+\beta+\gamma)$, the second last equation gives

$$\alpha^2 + 2S\gamma\alpha + \gamma^2 = 0, \quad \text{or} \quad (\alpha+\gamma)^2 = 0,$$

therefore $\gamma = -\alpha$, and similarly $\delta = -\beta$.

That the quadrilateral is a plane figure, is therefore evident, because $S\alpha\beta\gamma$ and $S\beta\gamma\delta$ are each equal to 0. It is of course a parallelogram, because $$T\gamma = T\alpha, \quad \text{and} \quad T\delta = T\beta.$$

II. *Solution by* ASHER B. EVANS, M.A.

Let A, B, C, D be the vertices of any quadrilateral; pass a plane through A, B, C and let D_1 be the projection of d on this plane.

If $(AD)^2 + (BC)^2 = (AB)^2 + (BD)^2 + (DC)^2 + (AC)^2$, we must have, since $(AD)^2 = (AD_1)^2 + (DD_1)^2$, $(BD)^2 = (BD_1)^2 + (DD_1)^2$, $(DC)^2 = (CD_1)^2 + (DD_1)^2$,

$$(AD_1)^2 + (BC)^2 - (DD_1)^2 = (AB)^2 + (BD_1)^2 + (CD_1)^2 + (AC)^2 \ldots\ldots (1).$$

Let X be the distance between the middle points of AD_1 and BC; then

$$(AD_1)^2 + (BC)^2 + 4X^2 = (AB)^2 + (BD_1)^2 + (CD_1)^2 + (AC)^2 \ldots\ldots (2).$$

From (1) and (2) we have $4X^2 + (DD_1)^2 = 0$, which can only be satisfied when $X=0$ and $DD_1=0$; but in this case D is in the plane passing through A, B, C, and AC and BD bisect each other; or ABCD is a plane figure and a parallelogram.

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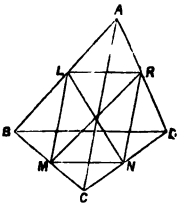
III. *Solution by* Professor WOLSTENHOLME, M.A.

If A, B, C, D be four points in space, and BA, BC, DA, DC be bisected in L, M, R, N; LM, NR are parallel to CA; LR, MN to BD; hence LMNR is a parallelogram, and therefore



$$NL^2 + MR^2 = 2LM^2 + 2MN^2,$$

or $$2NL^2 + 2MR^2 = AC^2 + BD^2.$$

Hence if aa', bb', cc' be the lengths of pairs of opposite edges, and x, y, z the lengths of the lines joining the middle points of the three pairs, we have

$$a^2 + a'^2 = 2(y^2 + z^2), \quad b^2 + b'^2 = 2(z^2 + x^2), \quad c^2 + c'^2 = 2(x^2 + y^2).$$

Hence if $b^2 + b'^2 + c^2 + c'^2 = a^2 + a'^2$, we have $x = 0$;

or if $$BC^2 + AD^2 + AB^2 + CD^2 = AC^2 + BD^2,$$

the middle points of AC, BD coincide, or ABCD is a parallelogram.

3962. (Proposed by J. J WALKER, M.A.)—If A, B, C, D be four points whose distances asunder are $BC = a$, $CA = b$, $AB = c$, $DA = a'$, $DB = b'$, $DC = c'$; prove that whether these points are in the same plane or are the corners of a tetrahedron, we shall have the relations

$$2aa' \cos(aa') = (b^2 + b'^2) - (c^2 + c'^2), \quad 2bb' \cos(bb') = (c^2 + c'^2) - (a^2 + a'^2),$$

$$2cc' \cos(cc') = (a^2 - a'^2) - (b^2 + b'^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

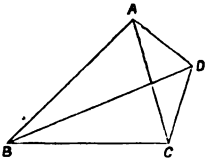
$$aa' \cos(aa') + bb' \cos(bb') + cc' \cos(cc') = 0 \quad \ldots\ldots\ldots\ldots\ldots (2),$$

and (3) if the four points are in the same plane, sixteen times the square of the area of the quadrilateral ABCD is equal to

$$(a^2 + a'^2 + 2bb' - c^2 - c'^2)(c^2 + c'^2 + 2bb' - a^2 - a'^2).$$

I. *Solution by* Professor WOLSTENHOLME, M.A.

1. If we project on BD the lines DA, AC, CB, we get $b' = a' \cos ADB + b \cos(bb') + a \cos CBD$



$$= a'\left(\frac{a'^2 + b'^2 - c^2}{2a'b'}\right) + b\cos(bb') + a\left(\frac{a^2 + b'^2 - c'^2}{2ab'}\right)$$

$$= \frac{(a^2 + a'^2) - (c^2 + c'^2)}{2b'} + b' + b\cos(bb'),$$

whence one of this group of expressions follows at once.

2. This expression is obtained by adding those in the first part.

3. The area of the quadrilateral is $\frac{1}{2}bb' \sin(bb')$, hence we have

$$16(\text{Area})^2 = 4b^2b'^2\left\{1 - \cos^2(bb')\right\} = 4b^2b'^2 - (c^2 + c'^2 - a^2 - a'^2)^2 = \&c.$$

II. *Solution by* R. F. SCOTT.

1. Let α, β, γ, δ be the vectors to the angular points A, B, C, D; and let γ be the angle between the diagonals; then

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$$2bb' \cos\phi = -2S.(\alpha-\gamma)(\beta-\delta) = -2S.(\alpha\beta-\beta\gamma-\alpha\delta+\gamma\delta)$$
$$= S\{(\alpha-\beta)^2-(\beta-\gamma)^2+(\gamma-\delta)^2-(\delta-\alpha)^2\} = (a^2+a'^2)-(c^2+c'^2).$$

2. This follows at once from

$$S.\{-(\alpha-\beta)(\gamma-\delta)+(\beta-\gamma)(\delta-\alpha)+(\alpha-\gamma)(\beta-\delta)\} \equiv 0.$$

3. $2(\text{Area}) = bb'm\phi$, therefore, &c.

4032. (Proposed by Prof. CROFTON, F.R.S.)—A random shot has struck a circular target (area Ω); the chance that it has fallen within a specified space on the target of any form (area ω) is of course $p = \frac{\omega}{\Omega}$; but if another random shot has struck the target *further* from the centre than the first, show that this additional knowledge alters the above probability to

$$p_1 = 2\frac{\omega}{\Omega} - \frac{m}{M},$$

where m, M are the moments of inertia of the areas ω, Ω round the centre of Ω. If the radii of gyration are equal, p is not altered. Also find p if the second shot be *nearer* instead of further.

I. *Solution by* G. S. CARR.

The number of possible positions of the two shots with the first nearest to the centre is $= \frac{1}{2}\pi r^2 . \pi r^2 = \frac{1}{2}\pi^2 r^4$, where r = radius of target.

Let $F(x)\,dx$ be an elemental strip of the area ω included between the circumferences whose radii are x and $x+dx$, so that $\omega = \int_k^k F(x)\,dx$.

Then the number of ways in which the first shot can fall within the area ω, the second being at the same time farther from the centre of the circle, is

$$\int_k^k F(x)\,dx\,\pi(r^2-x^2)$$
$$= \pi r^2 \int_k^k F(x)\,dx - \pi\int_k^k F(x)\,x^2\,dx = \pi r^2\omega - \pi m \ldots\ldots (1).$$

Thus
$$p_1 = \frac{\pi r^2\omega - \pi m}{\frac{1}{2}\pi^2 r^4} = \frac{2\omega}{\Omega} - \frac{m}{M}$$

as was to be shown.

Putting $m = \omega k^2$ and $M = \Omega K^2$, we see that when $k = K$, we have

$$p_1 = \frac{2\omega}{\Omega} - \frac{\omega}{\Omega} = \frac{\omega}{\Omega},$$

as in the first case.

If the second shot is known to be nearest to the centre, we have, instead of (1),

$$\int_k^k F(x)\,dx\,\pi x^2 = \pi m;$$

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therefore $$p_2 = \frac{\pi m}{\frac{1}{4}\pi^2 r^4} = \frac{m}{\mathrm{M}},$$

so that $p_1 + p_2 = \frac{2\omega}{\Omega} = 2p$, as it ought to be.

It may be worth observing that $\frac{1}{2}p_1$ is the *à priori* probability of the first shot falling within the area ω, and also nearer to the centre of the circle than the second shot; and p_1 is the probability of the same event when it is *certain* that the first shot is nearest to the centre.

II. *Solution by* A. M. Nash.

If a be the radius of the target, and r the distance of the first shot from the centre, the probability that the second shot should be further from the centre is $\frac{a^2 - r^2}{a^2}$; but since r may have any value within the limits of the area ω, we have

$$p_1 = \frac{\int_{r_1}\int_{\theta_1}^{\theta} (a^2 - r^2)\, r\,d\theta\,dr}{a^2 \int_{r_1}\int_{\theta_1}^{\theta_2} r\,d\theta\,dr} = \frac{a^2\omega - m}{a^2\omega} \quad \ldots\ldots\ldots\ldots (1);$$

and if p_2 be the probability that the second shot is further from the centre than the first, when the first hits any other part of the target, then

$$p_2 = 1 - \frac{\int_0^a \int_0^{2\pi} r^3\,d\theta\,dr - \int_{r_1}\int_{\theta_1}^{\theta_2} r^3\,d\theta\,dr}{a^2 \int_0^a \int_0^{2\pi} r\,d\theta\,dr - a^2 \int_0^a \int_0^{2\pi} r\,d\theta\,dr} = 1 - \frac{\mathrm{M} - m}{\frac{1}{2}(\Omega - \omega)} \ldots (2).$$

The *à priori* probabilities of (1) and (2) are $\frac{\omega}{\Omega}$ and $\frac{\Omega - \omega}{\Omega}$; therefore the required probability is

$$p = \frac{\frac{\omega}{\Omega}\,\frac{a^2\omega - m}{a^2\omega}}{\frac{\Omega - \omega}{\Omega}\left\{1 - \frac{\mathrm{M} - m}{a^2(\Omega - \omega)}\right\}} = \frac{\omega - \frac{m}{a^2}}{\Omega - \frac{m}{a^2}}.$$

But $\mathrm{M} = \frac{1}{2}a^2\Omega$, therefore $p = \frac{2\omega}{\Omega} - \frac{2m}{a^2\Omega} = \frac{2\omega}{\Omega} - \frac{m}{\mathrm{M}}$.

If the radii of gyration are equal,

$$\frac{m}{\mathrm{M}} = \frac{\omega}{\Omega}, \quad \text{therefore } p_1 = \frac{\omega}{\Omega}.$$

If the second shot be nearer than the first, we get, as before,

$$p_1 = \frac{m}{a^2\omega}, \quad p_2 = \frac{\mathrm{M} - m}{a^2(\Omega - \omega)}, \quad p = \frac{\frac{\omega}{\Omega}\cdot\frac{m}{a^2\omega}}{\frac{\omega}{\Omega}\cdot\frac{m}{a^2\omega} + \frac{\Omega - \omega}{\Omega}\cdot\frac{\mathrm{M} - m}{a^2(\Omega - \omega)}} = \frac{m}{\mathrm{M}}.$$

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3648. (Proposed by the EDITOR.)—Show (1) that the pedal of a four-cusped hypocycloid is a curve which may be defined by the corresponding equations in polar and rectangular coordinates

$$r = a \sin 2(\theta + \alpha) - c \cos \theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\text{or } (x^2+y^2+cx)^2(x^2+y^2) = a^2\left\{(x^2-y^2)\sin 2\alpha + 2xy\cos 2\alpha\right\}^2 \ldots (\alpha');$$

$2a$ being the radius of the fixed circle from which the hypocycloid is generated, c the distance of the pedal origin O from the centre Q, and α the inclination of QO to a cuspidal tangent; (2) that when the origin O is at the centre Q, the pedal is the four-branched *corolla*

$$r = a \sin 2\theta \ldots\ldots (\beta), \text{ or } (x^2+y)^3 = 4a^2x^2y^2 \ldots\ldots (\beta');$$

(3) that when the origin O is in a radius QA drawn to a vertex A of the hypocycloid, the pedal is the four-branched *scarabœus*

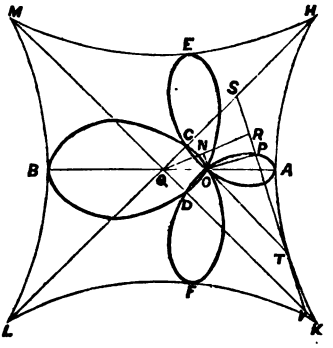
$$r = a\cos 2\theta - c\cos\theta \ldots (\gamma), \text{ or } (x^2+y^2+cx)^2(x^2+y^2) = a^2(x^2-y^2)^2 \ldots (\gamma');$$

and show further (4, 5) that when O is between Q and A, this pedal has four real branches passing through O and forming two unequal loops that touch two branches of the hypocycloid at opposite vertices (A, B), and two equal loops that touch the other two branches of the hypocycloid at points (E, F) on the same side of the vertices with the origin; but (6) if O is at the vertex A, two of the branches of the pedal become there a cusp which, near the origin, takes the form of the semi-cubical parabola $2x^3 + 3ay^2 = 0$; and (7) when O is outside the hypocycloid, the origin is a conjugate point through which only two real branches pass. Also show (8) that the area of any loop of the pedal may be obtained from the formula

$$\tfrac{1}{4}a^2\left\{(1+e^2)\theta - 3e\sin\theta(7-\cos^2\theta)\right\},$$

and that the entire area of the pedal is $\frac{1}{2}(1+e^2)\pi a^2$.

Solution by the PROPOSER.



1. Let HKLM be the four cusps of the hypocycloid, Q the centre of the figure, O the pedal origin, SV a line touching the curve in T and meeting the cuspidal tangents in S and V, and P the foot of the perpendicular from O on STV; then the locus of P will be the required pedal.

Join OT, and draw ON and QNR respectively parallel and perpendicular to STV. Let $QH = 2a$, $QO = c$, $\angle KQO = \alpha$, $OP = r$, $\angle AOP = \theta$; thus the pedal origin is defined by the polar coordinates (c, α), referred to QK as initial axis; and (r, θ) are the polar coordinates of P in reference to O as origin and OA as axis.

Now it is a well-known property of a quadrantal hypocycloid that the portion of a tangent intercepted between two cuspidal tangents is constant and equal to the radius QH of the fixed circle; thus in the figure $SV = QH = 2a$. Now we have

$$QS = SV\cos S = 2a\cos(\theta+\alpha),$$
$$QR = QS\sin S = a\sin 2(\theta+\alpha),$$
$$QN = QO\cos OQN = c\cos\theta, \text{ and } r = QR - QN;$$

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hence we see at once that the most general form of the polar equation of the pedal is that (α) given in the first part of the question, and the corresponding equation (α') in x and y is readily obtained therefrom.

2. When the pedal origin O is at the centre Q of the hypocycloid, we have $c=0$ and $a=0$, the axis being now taken along a cuspidal tangent; hence, in this case, the equation of the pedal takes the forms (β) or (β') given in the second part of the question, the latter (β') of which has been already given in an Editorial note on the solution of Question 3773, on p. 41 of Vol. XVIII. of the *Reprint*. The pedal here consists of four branches crossing each other at the origin and forming four equal and symmetrical loops which touch the hypocycloid at its vertices. On account of its fanciful resemblance to the petals of an open flower, this curve, whose general equation is $r = a \sin n\theta$, or $r = a \cos n\theta$, has been called the *corolla*. Several varieties of these curves are said (DAVIES' *Hutton*, Vol. ii. p. 347) to be figured by GUIDO GRANDI in a paper in the *Philosophical Transactions*, where they are quaintly designated "A bouquet of flowers for the Royal Society." The locus of the middle point of the straight line which joins the extremities of the hands of a clock, supposed of equal length a, is a *corolla*; its equation being $r = a \cos \frac{1}{3}\theta$. This forms the *second* part of the EDITOR'S Question 3682; of the *first* part of which (the envelope of the joining line or the first negative pedal of the corolla) a solution by Mr. LAVERTY has been given on p. 63 of Vol. XVII. of the *Reprint*.

3. When the pedal origin is between the centre and a vertex of the hypocycloid, we have in the equations of Art. 1, $\alpha = 45°$, and the pedal is then defined by either of the two equations (γ) or (γ') given in Art. 3 of the question. Put now e for $(c : a)$, the *excentricity* of the pedal, so that $c = ae$; then, from the first of these equations we find that r vanishes when

$$\cos\theta = \tfrac{1}{4}\left\{e \pm (e^2+8)^{\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\delta).$$

Moreover, by differentiating with respect to θ, and equating $dr : d\theta$ to zero, we find that, in absolute magnitude, r is a maximum when either $\sin\theta = 0$ or $\cos\theta = \frac{1}{4}e$, and that, disregarding signs, these maximum values are $r = (1-e)\,a = \text{OA}$, $r = (1+e)\,a = \text{OB}$, $r = (1+\frac{1}{8}e^2)\,a = \text{OE} = \text{OF}$.

The pedal is here of the form traced in the diagram, a curve which has been called a *scarabæus*. In order to obtain the geometrical genesis of the curve, let us suppose the point of contact T of the tangent to start from A and move round the hypocycloid in the direction AKLMH. Then, as T moves from A towards K, the pedal point P will move from A towards O, passing through O when $\theta = \cos^{-1}\left\{\frac{1}{4}e + \frac{1}{4}(e^2+8)^{\frac{1}{2}}\right\}$ cutting the axis there at this angle, and arriving at D, the foot of the perpendicular from O on QK, when T reaches K. As the tangent turns over the branch KL, P advances from D to a point F, where the hypocycloid and its pedal have a common tangent perpendicular to OF, when $\theta = \cos^{-1}(\frac{1}{4}e)$—in this figure nearly a right angle,—and r has the maximum value $(1+\frac{1}{8}e^2)\,a$. As T moves from F towards the left, P moves to the right; when T reaches the vertex of the branch KL, then $\theta = 90°$ and measured negatively downwards, r has the value a, having thus been diminished by $\frac{1}{8}ae^2$ from its value at F; when T arrives at a point where $\theta = \cos^{-1}\left\{\frac{1}{4}e - \frac{1}{4}(e^2+8)^{\frac{1}{2}}\right\}$, P passes through O at this inclination to the axis; when T reaches L, then P arrives at C, the foot of the perpendicular from O on QH; and as T moves over the arc LB, P traces out the part of the pedal from C to B. We have thus

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gone minutely over one half of the pedal, and the other half, traced by P as T moves over the arcs BMHA, is precisely symmetrical to this.

4. The equation in θ at the points of contact of the hypocycloid and its pedal may be readily obtained by eliminating between the equations of the two curves. The equation of the hypocycloid, in rectangular coordinates (x', y') along QH and QM as axes, is

$$(x')^{\frac{2}{3}} + (y')^{\frac{2}{3}} = (2a)^{\frac{2}{3}}, \text{ or } (4a^2 - x'^2 - y'^2)^3 = 27\,(2ax'y')^2 \ldots\ldots (\epsilon).$$

Now $x' = r\cos(\theta - 45°) + c\cos 45°$, and $y' = r\sin(\theta - 45°) - c\sin 45°$;

or $x'\sqrt{2} = r(\sin\theta + \cos\theta) + c$, and $y'\sqrt{2} = r(\sin\theta - \cos\theta) - c$;

therefore*

$$2x'y' = (r\sin\theta)^2 - (r\cos\theta + c)^2 = -a^2\cos 2\theta\left\{\cos^2 2\theta + e\sin^2\theta\,(4\cos\theta + e)\right\},$$

also†

$$x'^2 + y'^2 = a^2(\cos^2 2\theta + e^2\sin^2\theta);$$

hence, by substitution, the equation (δ) in θ at the points of meeting of the hypocycloid and its pedal, becomes

$$(4 - \cos^2 2\theta - e^2\sin^2\theta)^3 = 27\cos^2 2\theta\left\{\cos^2 2\theta + e\sin^2\theta\,(4\cos\theta - e)\right\}^2 \ldots (\zeta).$$

This equation (ϵ) is satisfied by

$$\sin^2\theta = 0, \text{ and by } 4\cos\theta - e = 0;$$

the first of which gives the points A and B, and the second the points E and F, where $\cos\theta = \frac{1}{4}$, as determined in Art. 3.

5. Taking now equation (γ'), put therein $y = mx$, and reject all powers of x above the first; then, for points near the origin O, the values of m will be given by the expression

$$m^2 = \frac{2a^2 + c^2 + 4cx \pm (c^4 + 8a^2c^2 + 32a^2cx)^{\frac{1}{2}}}{2\,(a^2 - 2cx)} \ldots\ldots\ldots\ldots (\eta).$$

Now the limit, when $x = 0$, the corresponding values of m will be those of $\frac{dy}{dx}$ at the origin O, and the number of such values will therefore determine the multiplicity of the point. Calling the values at the origin m_0, &c., we have from (η),

$$m_0^2 = \left(\frac{dy}{dx}\right)_0^2 = \tan^2\theta_0 = \tfrac{1}{2}\left\{e^2 + 2 \pm e\,(e^2 + 8)^{\frac{1}{2}}\right\} \ldots\ldots\ldots (\theta).$$

This result may be readily shown to agree with that in (δ); for, at the origin O, where $r = 0$, we have

$$\tan^2\theta_0 = \frac{1 - \cos 2\theta_0}{1 + \cos 2\theta_0} = \frac{1 - e\cos\theta_0}{1 + e\cos\theta_0} = \frac{4 - e^2 \mp e\,(e^2 + 8)^{\frac{1}{2}}}{4 + e^2 \pm e\,(e^2 + 8)^{\frac{1}{2}}},$$

which is equivalent to the result in (θ).

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* The interpretation of this transformation leads to the following geometrical theorem:—*Let* XEQ, YEQ *be two right angled triangles on the same hypotenuse* EQ, *such that* QX *and* QY *contain half a right-angle; then* $2XE.XQ = YE^2 - YQ^2$. For if X and Y be the projections of the point E upon QH and QA respectively, we have $XQ = x'$, $XE = y'$, $YQ = r\cos\theta + c$, $YE = r\sin\theta$; hence the above relation is the *algebraical* expression of this theorem. A *geometrical* proof may be readily obtained by drawing a semicircle through E, X, Y, Q; joining X and Y with the centre O; and drawing XM, YN perpendicular to EQ; then XOY is a right angle, and the triangles XMO, ONY are equal in all respects; therefore

$$2XE.XQ = 4\Delta XEQ = 2EQ.XM = 2EQ.ON = EN^2 - NQ^2 = YE^2 - YQ^2.$$

† This is an expression for the square of the distance QP from Q to a point (r, θ) on the pedal; for $QP^2 = QR^2 + RP^2$, also $QR = a\cos 2\theta$ and $RP = ON = c\cos\theta$.

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Now so long as $c < a$ or $e < 1$, the four values of m_0 are all real, and give the trigonometrical tangents of the angles at which the four real branches cut the axis at the origin O.

6. When $c = a$ or $e = 1$, the values of m_0 are $\pm\sqrt{3}$ and 0; which shows that when O is at A the head of the scarabæus vanishes, that two of the wing-branches (OCE, ODF) form a cusp to which the axis is a common tangent, and that each of the other two wing-branches (EOD, FOC) cuts the axis at an angle of 60°; also the maximum radii OE, OF are inclined to the axis at an angle of 75° 32′ ($= \cos^{-1}\frac{1}{4}$). Moreover, in this case, if we develope the right-hand side of equation (η), rejecting the second and all higher powers of x, and taking the lower sign, which applies to the wing-branches OCE, ODF, we have

$$m^2 = \frac{3}{2}\left(1-2\frac{x}{a}\right)^{-1}\left\{1+\frac{4}{3}\frac{x}{a}-\left(1+\frac{32}{9}\frac{x}{a}\right)^{\frac{1}{2}}\right\}$$

$$= \frac{3}{2}\left(1+2\frac{x}{a}+\ldots\right)\left(-\frac{4}{9}\frac{x}{a}+\ldots\right),$$

or $$\frac{y^2}{x^2} = -\frac{2}{3}\frac{x}{a}, \text{ or } y^2 = -\frac{2}{3}\frac{x^3}{a}, \text{ or } 2x^3+3ay^2 = 0 \quad \ldots\ldots\ldots (\iota);$$

hence, near the origin, the wing-branches OCE, ODF take the form of the semi-cubical parabola defined by equation (ι).

7. When $c > a$, or $e > 1$, two of the values of m_0 in (θ) are imaginary, and the other two real; hence in this case the two wing-branches OCE, ODF do not reach to the origin, but form a cusp at A; and the other two wing-branches EOD, FOC alone pass through the origin, and cut the axis there at an angle greater than 60°.

8. For the area of the pedal we have

$$\int \tfrac{1}{2}r^2 d\theta = \tfrac{1}{4}a^2 \int 2(\cos 2\theta - e\cos\theta)^2 d\theta$$

$$= \tfrac{1}{4}a^2\left\{(1+e^2)\theta - 2e\sin\theta + \tfrac{1}{2}e^2\sin 2\theta - \tfrac{2}{3}e\sin 3\theta + \tfrac{1}{4}\sin 4\theta\right\} \ldots (\kappa).$$

This formula will give the area of *any portion* of the curve; but since, when the curve passes through the origin, we have then $\cos 2\theta = e\cos\theta$, the area of *any loop* may be obtained from the formula

$$\tfrac{1}{4}a^2\left\{(1+e^2)\theta - 3e\sin\theta\,(7-\cos^2\theta)\right\} \ldots\ldots\ldots\ldots (\lambda),$$

which is what (κ) becomes under these conditions.

Taking (2) between the limits $(0, \pi)$ and doubling the result, we find the area of the whole curve to be

$$\tfrac{1}{2}(1+e^2)\pi a^2 \ldots\ldots\ldots\ldots (\mu),$$

which is half the area of the rolling circle that generates the hypocycloid together with half the circle whose radius is the distance of the pedal origin from the centre. This furnishes an illustration of the general theorem on pedal areas given in Question 1435, on p. 35 of Vol. I. of the *Reprint*.

3956. (Proposed by the Rev. Dr. Booth, F.R.S.)—Two planes are given in position, and a point equi-distant from them. Find the locus of

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a point whose distance from the former shall be equal to the square root of the product of the perpendiculars on the given planes from the moving point.

Solution by Professor TOWNSEND, F.R.S.

The equation $S = MN$, where S is any quadric and M and N any two planes real or imaginary, represents in general a second quadric S′, intersecting S in its conics of intersection real or imaginary with M and N, and having consequently double contact with it at the extremities real or imaginary of the chord MN. Hence when S is a sphere of any radius, the chord of double contact MN is parallel to an axis of S′, the cyclic axis when M and N are real; and when, as in the case proposed, it is a sphere of evanescent radius, its centre P is a focus of S′, of the umbilicar or modular species according as M and N are real or imaginary, its chord of double contact MN, the polar with respect to S′ of the focal tangent at P, being in either case the corresponding directrix. In the particular case proposed, denoting by l the length of the perpendicular PL from the given point P on the line of intersection L of the two given planes M and N, by ϕ the angle made by PL with either of those planes, by Q and R the points of meeting with M and N of the line through P perpendicular to the plane through P and L, by O the centre of the sphere touching M and N at Q and R, and by E and F its points of intersection with the line PL on which its centre O manifestly lies; the required locus is the ellipsoid whose centre is O, whose transverse axis is EF, whose mean axis is parallel to MN and equal to PQ, and whose conjugate axis is parallel to PQ and has to the transverse axis EF the ratio of PL to PL + OL, the interval OP being the semi-transverse axis of its focal hyperbola; the three semi-axes a, b, c of the surface, expressed in terms of l and ϕ, being $b = l \tan \phi$, $a = l\dfrac{\tan \phi}{\cos \phi}$, $c = l\dfrac{\tan \phi}{(1 + \cos^2 \phi)^{\frac{1}{2}}}$, respectively; and the ellipsoid itself opening out with an elliptic paraboloid of revolution, of which P is the focus and M or N the director plane, when $\phi = 0$, that is when M and N coincide.

[An exhaustive discussion of the general problem of which this is a particular case, with the history of the entire subject in connection with the theory of confocal quadrics, is given in SALMON's *Geometry of Three Dimensions*, 2nd edition, Chap. viii.]

4039. (Proposed by C. H. HINTON.)—On the portions of the sides of a triangle enclosed between the internal and external bisectors of the opposite angles respectively as bases, are constructed three rectangles of equal area; show that the side of one is equal to the sides of the other two.

Solution by the Rev. J. L. KITCHIN, M.A.

Let a, b, c be the sides of the triangle; then it is at once found that the

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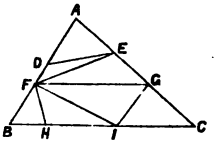
intercepts between the bisectors are $\frac{2abc}{b^2-a^2}$, $\frac{2abc}{c^2-b^2}$, $\frac{2abc}{a^2-c^2}$, where one of the latter, as written, must be negative.

Let the other sides of the rectangles, taken in order, be x, y, z; then we have $\frac{x}{b^2-a^2} = \frac{y}{c^2-b^2} = \frac{z}{a^2-c^2}$; therefore $x+y-z=0$ or $z=x+y$; and according as we take the sides we shall get $x=y+z$ or $y=x+z$.

3400. (Proposed by ARTEMAS MARTIN.)—Show that the average area of all the hexagons that can be formed by cutting off the corners of a given triangle, the points of section being uniformly distributed over the sides, is two-thirds of the area of the triangle.

Solution by the PROPOSER.

Let ABC be the given triangle; a, b, c the sides as usual; $u=$AD, $v=$BF, $w=$AE, $x=$GC, $y=$BH, $z=$IC, Δ = area of the given triangle, and M = average area required.



Then area of hexagon GIHDEF =

$$\Delta - \tfrac{1}{2}(uw \sin A + vy \sin B + xz \sin C) \quad \ldots (1);$$

and $$M = \frac{8}{a^2b^2c^2}\int_0^c\int_0^{c-w}\int_0^b\int_0^{b-v}\int_0^a\int_0^{a-y} (1)\, du\, dv\, dw\, dx\, dy\, dz$$

$$= \Delta - \tfrac{1}{8}(\tfrac{1}{2}bc \sin A + \tfrac{1}{2}ac \sin B + \tfrac{1}{2}ab \sin C) = \Delta - \tfrac{1}{8}(3\Delta) = \tfrac{5}{8}\Delta.$$

[Mr. MARTIN'S result may be obtained without integration, by the aid of the following simple geometrical considerations.

Supposing I to be the only variable point, the triangle FGI would have its mean value when I is the middle point of CH; since, for every position of I on one side of this middle point which gives an area above this mean, there is obviously another on the other side, giving an area just as much below it. Similarly, when I is a fixed and H a variable point, the triangle FHI will have its mean value when H is the middle point of BI. Hence, when both points vary in position, the quadrilateral FGIH will have its average value when H and I trisect BC; and similarly for D and F, and for E and G. Thus *the average hexagon required will have its vertices at the points of trisection of the sides of the triangle;* and since each of the triangles cut off will be one-ninth of the whole triangle, the area of the mean hexagon will be two-thirds of the triangle.

By similar reasoning we may show that the average *triangle* amongst all those that have their vertices equably distributed over the sides of a triangle, one on each, will have its vertices at the *middle points* of the sides of the triangle.

If the vertices of the hexagon be determined by drawing lines parallel to the sides of the triangle through points equably distributed over its surface, it is shown in the solution of Question 2318 (*Reprint*, Vol. IX., p. 41) that the average area of the hexagon will be three-fourths of the area of the triangle.]

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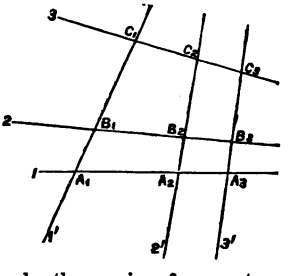
3910. (Proposed by F. D. Thomson, M.A.)—Show (1) that the three diagonals of any rectilinear hexagon drawn on the surface of an hyperboloid meet in a point; and (2) deduce therefrom Brianchon's Theorem for a plane conic.

I. *Solution by the* Proposer.



1. Let 1, 2, 3 be any three non-intersecting lines, and let 1′, 2′, 3′ be three lines drawn to meet them, and therefore generators of the hyperboloid through 1, 2, 3.

Then, with the notation of the figure, B_1A_2, C_1A_3, C_2B_3 are the common sections of the planes (11′, 22′), (33′, 11′), (22′, 33′); and therefore B_1A_2, C_1A_3, C_2B_3, the three diagonals of the hexagon formed by the six lines taken in the order 1, 2′, 3, 1′, 2, 3′, meet in a point.

Now any rectilinear hexagon on the hyperboloid must be formed in a similar way by three pairs of generators, hence the theorem of the question is true.

2. Brianchon's Theorem for a plane conic follows at once by orthogonally projecting the figure upon one of the principal planes of the hyperboloid. Or thus:—Join all the lines of the figure by planes to any point O in space. Then the six planes through O, and the six generators, are all tangent planes to the hyperboloid, and therefore to the tangent cone which has its vertex at O.

Now take any plane section of the cone and its six tangent planes, we then obtain a conic and a circumscribing hexagon with the three diagonals meeting in a point.

3. By taking the sides of the hexagon in different orders, we obtain in all, six distinct hexagons, for each of which the theorem is true.

Hence we have $9 + 6$ or 15 points arranged in space in sets of three on $6 + 6 \times 3$ or 24 straight lines.

II. *Solution by* Professor Townsend, F.R.S.

The two diagonals of the quadrilateral determined by any two pairs of generators of opposite systems, on a ruled quadric, being reciprocal polars to each other with respect to the surface, the three diagonals of the hexagon determined by any six (no three of which are collinear) of the nine intersections of any two triads of generators of opposite systems, pass consequently through the pole, with respect to the surface, of the plane determined by the remaining three; and therefore, &c., as regards the first part; from which the second follows at once by projection from any arbitrary point upon any arbitrary plane.

III. *Solution by* Professor Wolstenholme.

The hexagon being ABCDEF, take AB, BC, DE, EF as the edges of the tetrahedron of reference, whose equation may be taken $wx = yz$, the

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equations of the six sides being

$$(1)\ldots(w=0, z=0),\quad (2)\ldots(x=0, z=0),\quad (3)\ldots(x=z, w=y),$$
$$(4)\ldots(w=0, y=0),\quad (5)\ldots(x=0, z=0),\text{ and }(6)\ldots(x=y, w=z).$$

The equations of the diagonal (6 1), (3 4) are $w=0,\quad -x+y+z=0,$

„ „ (1 2), (4 5) „ $w=0,\quad x=0,$

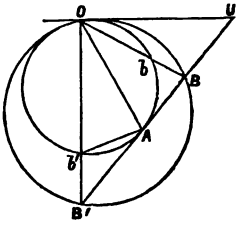
„ „ (2 3), (5 6) „ $x=0,\quad -w+y+z=0,$

whence all three meet in the point $w=0,\ x=0,\ y+z=0$.

Now project the whole system of the generating lines of both systems from any point on any plane whatever, the plane through the vertex and any generating line will be a tangent plane to the hyperboloid at some point lying on the polar plane of the point. Hence the projection of any generating line will be a tangent line to the projection of the curve of contact, and the hexagon will be projected into a Brianchon's Hexagon, the three diagonals of which are the projections of the diagonals of the original hexagon. Therefore, &c.

3834. (Proposed by J. J. Walker, M.A.)—Show (1) that any two isochronous straight lines of descent from a given point above an inclined plane to the plane are equally inclined to that of quickest descent (all being drawn in the same vertical plane); and (2) that the velocity acquired in sliding down the last is a mean proportional between the velocities acquired in sliding down the first-named lines.

Solution by Professor Wolstenholme, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*



1. If O be the given point, and a circle be described with O as highest point, meeting the inclined plane in B, B′, the times down OB, OB′ are equal, and the shortest time will be when B, B′ coincide (at A suppose). Let OB, OB′ meet the other circle in b, b'; then if OU be the horizontal line through O,

$$\angle UOB = \angle OB'B,\quad \angle UOA = \angle Ob'A,$$

therefore $\angle BOA = \angle B'Ab' = \angle AOB'$,

or OB, OB′ are equally inclined to OA.

2. The vertical depths of B, A, B′ below O are as UB : UA : UB′, but UB, UB′ = $UO^2 = UA^2$, hence the velocities acquired in falling down OB, OA, OB′ are in geometrical progression.

4060. (Proposed by the Rev. W. H. Laverty, M.A.)—The lengths of the seasons being known, find the eccentricity of the ecliptic.

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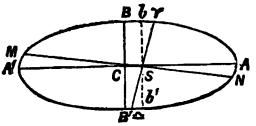
Solution by the PROPOSER.

Let AA′ be the apse line; ♈ ♎ the line of equinoxes; MN the line of solstices; CA = a, CB = b, CS = c. Now equal areas are described in equal times; therefore



$$\frac{\text{area S}\,\Upsilon\,\text{MS}}{\text{time of spring}} = \frac{\text{area SM}\,\simeq\,\text{S}}{\text{time of summer}} = \frac{\text{area S}\,\Upsilon\,\text{NS}}{\text{time of autumn}} = \frac{\text{area SN}\,\simeq\,\text{S}}{\text{time of winter}};$$

therefore $$\frac{\frac{1}{4}(\pi ab) + \text{S}b\,\Upsilon - \text{SMA}' + bc}{92\frac{7}{8}} = \frac{\frac{1}{4}(\pi ab) + \text{SMA}' - \text{S}b'\,\simeq + bc}{93\frac{7}{12}}$$

$$= \frac{\frac{1}{4}(\pi ab) + \text{S}b'\,\simeq - \text{SNA} - bc}{89\frac{71}{96}} = \frac{\frac{1}{4}(\pi ab) + \text{SNA} - \text{S}b\,\Upsilon - bc}{89\frac{1}{24}};$$

$$= \frac{\frac{1}{2}(\pi ab) + 2bc}{186\frac{11}{24}} = \frac{\frac{1}{2}(\pi ab) - 2bc}{178\frac{75}{96}};$$

(by adding together the numerators and denominators of the first two and of the last two fractions, and noticing that $\text{S}b\,\Upsilon = \text{S}b'\,\simeq$.)

Whence we get the eccentricity $= \frac{c}{a} = \frac{737 \cdot \pi}{4 \times 35363} = \cdot 0165$ nearly.

4064. (Proposed by Sir JAMES COCKLE, F.R.S.)—Notation and reference being as in Question 3874, let there be given

$$x^4 r = (x^3 + 2xy)\,p - 4y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now BOOLE (p. 220) in effect states that $y = cx^2$ is the singular solution of (1). Test this statement.

Solution by the PROPOSER.

Replacing Boole's arbitrary constants c and c' by a and $\log b$ respectively, the complete integral of (1) may, after a transformation, be written in the forms $\tan \log \left(\frac{x}{b}\right)^{\alpha} = \frac{y - x^2}{ax^2}$ or $\left(\frac{x}{b}\right) = \frac{y - (1+a)\,x^2}{y - (1-a)\,x^2}$.

In the first form put $b = 0$ or ∞. Then we have $y = (1 + a \tan \mp \infty)\,x^2$. Hence, whether we regard $\tan \infty$ as indeterminate, or whether, with De Morgan ("On Divergent Series," &c., *Camb. Trans.*, vol. viii., pt. II., §§ 3 and 4), we regard it as $\pm\sqrt{(-1)}$, we have $y = cx^2$, where c is arbitrary. Again, in the second form, put $b = \infty$. Then we have $y = (1 + a)\,x^2$, which may be written $y = cx^2$, where c is arbitrary. From either form I infer that $y = cx^2$ is a particular integral and not the singular solution of (1).

BOOLE (*Supplement*, p. 55), speaking of Legendre's tests for differential equations of the higher orders, says that the completion of the theory would consist in the discovery of those further tests dependent upon integration which correspond to the test of Euler and Cauchy for differential equations of the first order. Boole does not appear to have seen De Morgan's paper "On some points in the Theory of Differential Equations,"

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(*Camb. Trans.*, vol. ix. pt. IV.) De Morgan (§ 9) has extended Cauchy's theorem to biordinals thus. If $y = \mathrm{P}$ be a solution of $r = \chi(x, y, p)$, P being a function of x, the solution is singular or particular according as

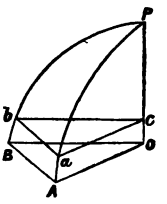
$$\int_0 dz \div \left\{ \chi\left(x, \mathrm{P}, \frac{d\mathrm{P}}{dx} + z\right) - \chi\left(x, \mathrm{P}, \frac{d\mathrm{P}}{dx}\right) \right\}$$

begins finite or infinite, z being the only variable.

Applying De Morgan's extension to the present question, the test function is reducible to $\log z$ which is infinite when $z = 0$. Hence $y = cx^2$ is a particular integral. Applying it to Question 3874, the test function is reducible to $\frac{1}{z}$, which is infinite when $z = 0$. Hence $y = cx$ is a particular integral. These results agree with those otherwise obtained. De Morgan (*ib.* § 10) points out that Cauchy's theorem may be extended to triordinals.

4023. (Proposed by W. H. H. Hudson, M.A.)—A series of strings knotted in squares like a rope-ladder form a belt round a sphere of radius r; if $r\theta$ be the length of a side, n the number of squares to reach right round, prove that $\tan\frac{\theta}{2} = \sin\frac{\pi}{n}$.

Solution by the Rev. G. H. Hopkins, M.A.



It is obvious that the sides of the squares which cut the equators of the sphere, will cut it so that they are parts of great circles passing through the poles, and be bisected by the equator.

The other sides will also lie on great circles, and will subtend at the centre an angle θ, and the chord ba subtends at c an angle equal to the angle BOA or $\frac{2\pi}{n}$.

$$\text{Chord } ba = 2\mathrm{R}\sin\tfrac{1}{2}\theta.$$

Also $$ba = 2bc\sin\frac{bCa}{2} = 2\mathrm{R}\cos\frac{\theta}{2}\sin\frac{\pi}{n};$$

therefore $$\sin\frac{\theta}{2} \propto \cos\frac{\theta}{2}\sin\frac{\pi}{n}, \quad \text{therefore } \tan\frac{\theta}{2} = \sin\frac{\pi}{n}.$$

4072. (Proposed by the Editor.)—If in the equations

$v + x + y + z = a$, $v^2 + x^2 + y^2 + z^2 = b$, $v^3 + x^3 + y^3 + z^3 = c$, $v^4 + x^4 + y^4 + z^4 = d$,

we have the relation $a^3 - 6ab + 8c = 0$; show (1) that the values of v, x, y, z will be one-fourth of the four values of the expression

$$a \pm \left\{ 4b - a^2 \pm 2\left(a^4 - 4a^2b - 4b^2 + 16d\right)^{\frac{1}{2}} \right\}^{\frac{1}{2}};$$

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and hence show (2) that if $a=20$, $b=150$, $c=1250$, $d=10674$, the values of v, x, y, z will be 1, 2, 8, 9.

Solution by the Rev. Robert Harley, F.R.S.

Employing the usual notation, we have

$$\Sigma v = a,\ \Sigma vx = \tfrac{1}{2}\left\{(\Sigma v)^2 - \Sigma v^2\right\} = \tfrac{1}{2}(a^2-b),$$

$$\Sigma vxy = \tfrac{1}{6}\left\{(\Sigma v)^3 - 3\Sigma v \Sigma v^2 + 2\Sigma v^3\right\} = \tfrac{1}{6}(a^3-3ab+2c),$$

$$vxyz = \tfrac{1}{24}\left\{\Sigma v^4 - 6(\Sigma v)^2 \Sigma v^2 + 8\Sigma v \Sigma v^3 + 3(\Sigma v^2)^2 - 6\Sigma v^4\right\}$$
$$= \tfrac{1}{24}(a^4 - 6a^2b + 8ac + 3b^2 - 6d).$$

We may therefore regard v, x, y, z as the roots of the quartic equation $V^4 - aV^3 + \frac{1}{2}(a^2-b)V^2 - \frac{1}{6}(a^3-3ab+2c)V + \frac{1}{24}(a^4-6a^2b+8ac+3b^2-6d) = 0$, and since $a^3-6ab+8c=0$, or what is the same thing, $c = -\frac{1}{8}(a^3-6ab)$, this equation may be written

$$V^4 - aV^3 + \tfrac{1}{2}(a^2-b)V^2 - \tfrac{1}{8}(a^3-2ab)V + \tfrac{1}{8}(b^2-2d) = 0.$$

Put $V = W + \frac{1}{4}a$, then the transformed equation is

$$W^4 + \tfrac{1}{8}(a^2-4b)W^2 - \tfrac{1}{256}(3a^4-8a^2b-32b^2+64d) = 0,$$

the solution of which is

$$W = \pm\tfrac{1}{4}\left\{-a^2+4b \pm 2(a^4-4a^2b-4b^2+16d)^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

and therefore

$$4V = a \pm \left\{-a^2+4b \pm 2(a^4-4a^2b-4b^2+16d)^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

In the particular case, $a^3-6ab+8c = 8000-18000+1000 = 0$, and therefore $4V = 20 \pm \left\{200 + 2(784)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = 20 \pm (200 \pm 56)^{\frac{1}{2}} = 36, 4, 32, 8$; therefore $V = 9, 1, 8, 2$.

3971. (Proposed by C. Taylor, M.A.)—The diagonals of a quadrilateral intersect at right angles in O. Show how to inscribe a conic with O as focus.

Solution by Professor Wolstenholme, M.A.

Since any rectangle has its corners on a circle, we see by reciprocation that the sides of a quadrilateral whose diagonals intersect at right angles in O will touch a conic, focus O. Let A, B, C, D be the corners of the quadrilateral; Oa, Ob, Oc, Od perpendiculars on BC, CD, DA, AB; then the perpendicular from C on ab will pass through the second focus of the conic, and also that from D on bc, and drawing these the second focus is found, and since a, b, c, d lie on the auxiliary circle, the conic is completely determined.

In exactly the same way in three dimensions a sphere circumscribes any rectangular parallelepiped; hence reciprocating, if we have an octahedron

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with triangular faces whose three diagonals intersect at right angles in a point O, a prolate spheroid (or hyperboloid) can be described with one focus at O touching the eight faces, and if perpendiculars be let fall from O on the faces, the feet of these perpendiculars all lie on a sphere; also if from any summit of the octohedron a perpendicular be let fall on the plane through the feet of the perpendiculars from O on the faces belonging to that summit, this perpendicular will pass through the second focus of the spheroid, which can therefore be constructed immediately.

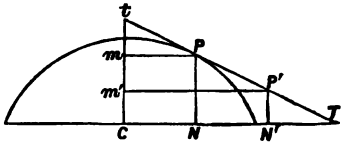
4062. (Proposed by R. Tucker, M.A.)—Give geometrical proofs of the tangential equations (in Cartesian coordinates) for the conic sections.

Solution by the Proposer.

1. For a central conic we have,

$$\frac{CN'}{CT} = \frac{P'M'}{CT} = \frac{O't}{Tt},$$

$$\frac{Cm'}{Ct} = \frac{P'N'}{Ct} = \frac{P'T}{Tt};$$

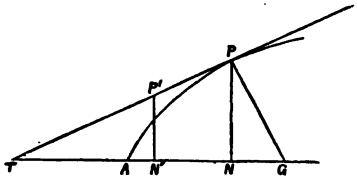


therefore $\frac{CN'}{CT} + \frac{CM'}{Ct} = 1$ or $\frac{CN \cdot CN'}{AC^2} + \frac{CM \cdot CM'}{CB^2} = 1,$

that is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$

2. For a parabola take P′ any point on the tangent, and P′N′ its ordinate, and draw the normal PG; then the triangle P′N′T, PNG are similar; therefore

$$\frac{P'N'}{N'T} = \frac{NG}{PN};$$



i. e., $P'N' \cdot PN = NG \cdot N'T,$

or $yy' = 2a(x + x').$

3922. (Proposed by A. Martin.)—Show (1) that the average area of all the triangles whose bases $= a$, and sum of the other two sides $= b$, is $\frac{1}{16}\pi a (b^2 - a^2)^{\frac{1}{2}}$; and (2) that the average area of the circles inscribed in them is $\frac{\pi a^2}{6}\left(\frac{b-a}{b+a}\right)$.

Solution by A. B. Evans, M.A.; *the* Proposer; *and others.*

1. Let a, x, $b - x$ represent the sides of the triangle, and r the radius of

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the inscribed circle. Then the area of the triangle is

$$\tfrac{1}{2}r(a+b) = \tfrac{1}{2}(b^2-a^2)^{\frac{1}{2}}\left\{a\left(\frac{a+b}{2}-x\right)-\left(\frac{a+b}{2}-x\right)^2\right\}^{\frac{1}{2}}.$$

As $x < \frac{1}{2}(b+a)$ and $> \frac{1}{2}(b-a)$, the number of different triangles formed under the given conditions is $\frac{1}{2}(b+a)-\frac{1}{2}(a+a) = a$, and the average is

$$\frac{1}{a}\int_{\frac{1}{2}(b-a)}^{\frac{1}{2}(b+a)}\left(\frac{b^2-a^2}{4}\right)^{\frac{1}{2}}\left\{a\left(\frac{b+a}{2}-x\right)-\left(\frac{b+a}{2}-x\right)^2\right\}^{\frac{1}{2}}dx = \frac{\pi a}{16}(b^2-a^2)^{\frac{1}{2}}.$$

2. The area of the inscribed circle is

$$\pi r^2 = \pi\left(\frac{b-a}{b+a}\right)\left\{a\left(\frac{b+a}{2}-x\right)-\left(\frac{b+a}{2}-x\right)^2\right\};$$

and therefore the average area is

$$\frac{1}{a}\int_{\frac{1}{2}(b-a)}^{\frac{1}{2}(b+a)}\left(\frac{b-a}{b+a}\right)\pi\left\{a\left(\frac{b+a}{2}-x\right)-\left(\frac{b+a}{2}-x\right)^2\right\}dx = \frac{\pi a^2}{6}\left(\frac{b-a}{b+a}\right).$$

[The *first* part of the Question is solved as part (3) of the Editor's Question 2360, on p. 103 of Vol. XIV. of the *Reprint*.

In the *second* part, the integration required to find the average is best effected by putting $\frac{1}{2}(b+a)-x = az$; for it then becomes

$$\pi a^2\left(\frac{b-a}{b+a}\right)\int_0^1 (z-z^2)\,dz = \frac{\pi a^2}{6}\left(\frac{b-a}{b+a}\right).]$$

3143. (Proposed by T. P. Kirkman, M.A., F.R.S.)—

Soon after Easter,
As every feaster
In Parliament can tell,
To every and to all his well-
beloved Peers and Commons
Mr. Punch presents
His compliments
And customary summons.

He admits to his levée and breakfast-
board
Along with the Ministers and P.C.'s,
Who go, of course, whenever they
please,
The other Senators in threes,
Three daily for just eight weeks;
And you may imagine how every Lord
And rising Commoner keenly seeks,
By hook or crook, to beg or buy ad-
mission in the morning triad.

"He has been thrice,
And I but twice?"—
Great is the din;
Teazing the Chancellor,—
Mobbing the Speaker,—
What name to cancel, or
Which to put in;
Who for next week, or
Who's down for to-morrow;
Whom to leave out
For a year to pout
In envious sorrow.

The Chancellor trims the triplets so,
While Peers implore,
And Commons roar,
That never a pair meet twice at the feast;
That all who go,
Go thrice at least,
And no one gets more
Than breakfasts four.

You have seen Mr. Punch
At dinner and lunch:
This yarn of his breakfast,
Will soon have your neck fast,
Unless you can check fast.

The fifty-six threes
Are formed without ease.
Can you do it for me,
And supply the key
Mathematically?
You've data enough,
If *quantum suff.*
Of other stuff.

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Solution by the PROPOSER.

AQR	SFH	NTP	ILE	
ASN	QIG	RUV	FEK	
GDM	VBJ	UCK		
HDJ	TBL	PCM		
A*ab*	B*ac*	C*ad*	Vγl	Sγj
A*ef*	B*gh*	C*ij*	V*uq*	S*wg*
G$s\gamma$	M*vg*	R*vp*	E*vz*	
G*cn*	M*fu*	R$m\beta$	E*lb*	
L*tf*	U*tn*	H*ty*	K*hx*	
L$j\beta$	U*kw*	H*pd*	K*se*	
N*xq*	D*x*β	Q*ia*	T*ka*	
N*zn*	DδR	Q*yh*	T*ps*	
F*ua*	J*re*	P*rm*	I*rw*	
F*cm*	J*zi*	P*ly*	I*qd*.	

Every capital occurs four times and every small letter thrice, and no duad is repeated.

It is very difficult to find an algebraic key to these triplets. We have $3 \cdot 56 = 3 \cdot 28 + 4 \cdot 21$. Twenty-one Peers, twenty-eight Commoners.

4041. (Proposed by W. S. BURNSIDE, M.A.)—In the analysis of the fundamental formulæ for the addition and subtraction of elliptic functions, prove geometrically that the following transformations are legitimate; viz., first change k to k^{-1}, and then change $\sin(amu)$ into $k \sin(amu)$.

Solution by G. M. MINCHIN, M.A.

Take a spherical triangle whose sides, BC, CA, AB are respectively ϕ, ψ, σ, or amu, amv, and $am(u+v)$. Then it is evident that we shall have $\cos A = \Delta(\phi)$, $\cos B = \Delta(\psi)$, $\cos C = -\Delta(\sigma)$,

and therefore $\sin A = k \sin\phi = k \sin a$.

Again take the polar triangle A′B′C′, and we shall have the derived modulus $= \dfrac{\sin A'}{\sin a'} = \dfrac{\sin a}{\sin A} = \dfrac{1}{k}$. Also $\sin a' = \sin A = k \sin\phi$.

This last equation is of course $\sin am\left(ku, \dfrac{1}{k}\right) = k \sin amu$.

4044. (Proposed by T. T. WILKINSON, F.R.A.S.)—From the foot of one of the perpendiculars of a triangle draw straight lines to the other two

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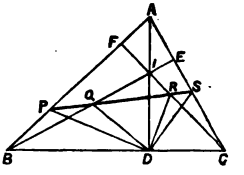
perpendiculars and also to the other two sides so as to make equal angles with these four lines all on the same side; then prove that the four intersections lie on the same straight line.

Solution by A. RENSHAW; S. WATSON; *the* PROPOSER; *and many others.*

Let AD, BE, CF be the perpendiculars upon the sides, intersecting in I; and from D draw DP, DQ, DR, DS to meet AB, BE, CF, CA in P, Q, R, S respectively, and making

$$\angle DPB = \angle DQB = \angle DRF = \angle DSA.$$



Then, because of the equality of those angles, a circle will pass through each of the four points B, P, Q, D; D, Q, I, R; hence

$$\angle DQP = 180° - B, \text{ and } \angle DQR = \angle DIR = B,$$

therefore $$DQP + \angle DQR = 180°.$$

Similarly $$\angle DRS + \angle DRQ = 180°;$$

therefore the points P, Q, R, S all lie in the same straight line.

The proof is exactly similar when the four lines are drawn from either E or F.

[This is a particular case of the following more general theorem: If from the common intersection of the four circles circumscribing the four triangles of a complete quadrilateral, lines are inflected to the sides of the quadrilateral so as to make equal angles with these sides, the four intersections are in a straight line; a solution of which appeared in the *Diary* for 1853.]

4010. (Proposed by Professor CLIFFORD.)—Prove that the lines of curvature of a quadric surface are projected from an umbilic on a plane parallel to its tangent plane into a series of confocal Cartesian ovals.

I. *Solution by* J. J. WALKER, M.A.

Consider the central quadric

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

By a well-known theorem, due to the late Mr. R. L. Ellis, any line of curvature may be supposed to be determined by the intersection of the cone

$$\frac{l^2x^2}{fa^2} + \frac{my^2}{(f+h)b^2} + \frac{n^2z^2}{hc^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $l^2 = b^2 - c^2$, $m^2 = a^2 - c^2$, $n^2 = a^2 - b^2$, and consequently $l^2 + n^2 = m^2$. Taking an umbilic, in the plane of (ac), as the new origin, and as axes the semi-diameter (a') through this, the tangent to the principal section, and the tangent perpendicular to it, the equation of the cone, having its vertex

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at the umbilic and passing through the intersection of (1) and (2), will be

$$A\left(\frac{y^2+z^2}{b^2}-\frac{x^2}{a'^2}\right)^2+4D\frac{zx}{a'b}\left(\frac{y^2+z^2}{b^2}-\frac{x^2}{a'^2}\right)+\frac{4x^2}{a'^2}\left(\frac{By^2}{b^2}+\frac{Cz^2}{a'^2}\right)=0\ldots(3),$$

where $A=\frac{l^2n^2(f+h)}{m^2fh}$, $B=\frac{m^2}{f+h}$, $C=\frac{n^4f+l^4h}{m^2fh}$, $D=\frac{ln(n^2f-l^2h)}{m^2fh}$.

The intersection of this with the central section, parallel to the tangent plane at the umbilic, is found by making $x=a'$; viz.,

$$A(y^2+z^2)^2+4Db(y^2+z^2)z+2(2B-A)b^2(y^2+z^2)+4(C-B)b^2z^2 -4Db^3z+Ab^4=0\ldots..(4),$$

which is the equation to a Cartesian oval $\mu\rho-\mu'\rho'=\sigma$ .....(5),

ρ being the vector from the centre of the quadric as focus, ρ' that from a second focus situate on the semi-diameter of the section (ac) conjugate to a', at a distance (d) from the centre determined by $d=\frac{n^2-l^2}{ln}b$; as may be proved by identifying with (4) the equation to the oval (5) in y and z; viz.,

$$(\mu^2-\mu'^2)^2(y^2+z^2)^2-4\mu'^2(\mu^2-\mu'^2)d(y^2+z^2)z$$
$$-2\left\{\mu'^2(\mu^2-\mu'^2)d^2+(\mu^2+\mu'^2)\sigma^2\right\}(y^2+z^2)+4\mu'^4d^2z^2$$
$$+4\mu'^2d(\mu'^2d^2-\sigma^2)z+(\mu'^2d^2-\sigma^2)^2=0.$$

II. *Solution by* Professor Townsend.

This pretty property may be established geometrically, on well known principles, as follows:—

The projection W, upon any plane L, of the quartic of intersection UV of any two quadric surfaces U and V, from any point P on either of them U, is in general a binodal quartic, whose nodal chord EF is the intersection with L of the tangent plane T to U at P, whose two nodal points E and F are the intersections with L of the two generators M and N, real or imaginary, of U through P, and whose four nodal tangents EH and EH′, FK and FK′ are the intersections with L of the four tangent planes to U at the four intersections of M and N with V.

When M and N are tangents to V, and therefore to the conic TV and to the quartic UV, the two pairs of nodal tangents EH and EH′, FK and FK′ coincide, the two nodal points E and F become in consequence cusps, and the binodal quartic W becomes accordingly bicuspidal. In the same case the eight planes connecting M and N with the four vertices A, B, C, D of the tetrahedron self-reciprocal to U and V intersect evidently with L in eight tangents to W, four passing through E and four through F, and the four through E intersecting with the corresponding four through F at the four projections A′, B′, C′, D′ of A, B, C, D on L.

Hence, when P is an umbilic of U, in which case M and N are the two asymptotes of its evanescent circle of intersection with T; when in addition L is parallel to T, in which case E and F are the two circular points I and J at infinity on L; and when, in fine, V is confocal with U, in which case P is a focus of, and M and N consequently tangents to, the conic TV, while of the four lines PA, PB, PC, PD three are complanar with and pass through the remaining three umbilici Q, R, S of U; then is W a

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Cartesian oval whose three collinear foci are the three projections Q', R', S' of Q, R, S on L; and therefore, &c.

That confocal Cartesian ovals consist of two systems orthogonal to each other has been shown by Professor CROFTON (See *Proceedings of the London Mathematical Society*, Vol. I., No. 6). And that the two systems of lines of curvature of a convex quadric project stereographically from an umbilic upon any parallel to its tangent plane into two systems of orthogonal curves may be proved *à priori* as follows: Every pair of conjugate diameters, and therefore the two axes, of any section of the surface by a plane parallel and infinitely near to the tangent plane at any point, project evidently into a pair of conjugate diameters of the projection of the section, but the projection, from an umbilic upon any parallel to the umbilical tangent plane, of any plane section of the surface, being necessarily a circle, therefore, &c.

3816. (Proposed by N'IMPORTE.)—Given the point of contact of one of two fixed tangents to a parabola; show that the locus of the focus is a circle passing through the intersection of the tangents, and touching one of them.

Solution by C. LEUDESDORF.

The equation to the parabola may be taken as (supposing b variable)

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

To find the focus, put $y = -x(\cos\omega + i\sin\omega) + c$ ..... (2)

in the equation (1), then the condition that (2) should touch (1) is

$$\left(1 - \frac{c}{b}\right)\left(\frac{1}{a} + \frac{\cos\omega + i\sin\omega}{b}\right) = \frac{1}{a}, \quad \text{therefore} \quad c = \frac{ab(\cos\omega + i\sin\omega)}{b + a\cos\omega + ai\sin\omega};$$

therefore $$y + x(\cos\omega + i\sin\omega) = \frac{ab(\cos\omega + i\sin\omega)(b + a\cos\omega - ai\sin\omega)}{a^2 + 2ab\cos\omega + b^2}.$$

Equating real and imaginary terms on each side, we have

$$y + x\cos\omega = \frac{ab(a + b\cos\omega)}{a^2 + 2ab\cos\omega + b^2}, \quad x\sin\omega = \frac{ab(b\sin\omega)}{a^2 + 2ab\cos\omega + b^2};$$

therefore the focus is given by

$$\frac{x}{b} = \frac{y}{a} = \frac{ab}{a^2 + b^2 + 2ab\cos\omega};$$

therefore $$(a^2 + b^2 + 2ab\cos\omega)x = ab^2, \quad \text{and} \quad b = \frac{ax}{y};$$

therefore $x^2 + y^2 + 2xy\cos\omega = ax$, which proves the theorem.

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4057. (Proposed by J. H. Cotterill, M.A.)—A circle, the centre of which is C, rolls on a fixed circle, the centre of which is O; show that the centre of curvature of the curve traced by any point P rigidly connected with the rolling circle, is given by the following construction:—Through the point of contact (T) of the circles draw TK perpendicular to the normal PT, to meet CP produced in the point K, then OK intersects PT produced in the required centre.

I. *Solution by* B. Williamson, M.A.

Let Q be the centre of curvature of the roulette in question, and let $\theta = \angle CTK$; then by the expression for the radius of curvature of a roulette given in *Liouville's Journal*, Vol. x., by M. Transon (See Salmon's *Higher Plane Curves*, Art. 312, 2nd edition), we have

$$\frac{PQ}{TQ \cdot TP} = \frac{1}{\sin\theta}\left(\frac{1}{OT} + \frac{1}{CT}\right) = \frac{1}{\sin\theta}\frac{CO}{TO \cdot CT};$$

therefore

$$\frac{PQ}{TQ} = \frac{CO}{TO} \cdot \frac{TP}{CT \sin\theta} = \frac{CO}{TO} \cdot \frac{PK}{CK}.$$

Hence, by the well known property of transversals, the points O, Q, K are in a straight line.

II. *Solution by the* Proposer.

For let $OT = R$, $CT = r$, $\angle CTP = \theta$, and draw OS, CZ perpendiculars on PT; then, by similar triangles

$$\frac{KT}{TF} = \frac{OS}{FS} = \frac{R\sin\theta}{R\cos\theta - TF}$$

and

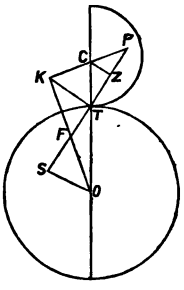
$$\frac{KT}{PT} = \frac{CZ}{PZ} = \frac{r\sin\theta}{PT - r\cos\theta};$$

therefore

$$\frac{PT}{TF} = \frac{R}{r} \cdot \frac{PT - r\cos\theta}{R\cos\theta - TF},$$

or

$$\cos\theta = \frac{R+r}{Rr} \cdot \frac{PT \cdot TF}{PF}.$$



Now imagine the rolling circle to move with uniform angular velocity ω; then P's acceleration may be considered as compounded of an acceleration ω^2. CP towards C, together with C's acceleration towards O, which last is evidently $\frac{r^2}{(R+r)^2}\omega^2 \cdot (R+r)$. Hence, resolving these two accelerations in the direction PT, we get

$$\text{P's normal acceleration} = \left(PZ + \frac{r^2}{R+r} \cdot \cos\theta\right)\omega^2.$$

But

$$PZ = PT - v\cos\theta,$$

therefore

$$\text{P's normal acceleration} = \left(PT - \frac{Rr}{R+r}\cos\theta\right)\omega^2$$

$$= \left(PT - \frac{PT \cdot TF}{PF}\right)\omega^2 = \frac{PT^2}{PF}\omega^2.$$

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Now P's velocity is PT . ω and its normal acceleration is consequently $\frac{PT^2}{\rho} . \omega^2$, where ρ is the radius of curvature of its path; whence evidently $\rho = PF$, that is, F is the required centre of curvature.

In the particular case of the epicycloid, this construction is given (in a slightly modified form) in Belanger's *Kinématique*, and is easily deducible from a construction given by Professor Willis, in his work on *Mechanism*.

4034. (Proposed by Professor CLIFFORD.)—Prove that the forty umbilics of a cubic surface which passes once, or of a quartic surface which passes twice, through the imaginary circle at infinity, lie by fives upon sixteen straight lines.

I. *Solution by* SAMUEL ROBERTS, M.A.

A quartic surface having the imaginary circle at infinity for a nodal line, contains 16 right lines, which meet the imaginary circle, or are "lines of no length." Each of these is intersected by 5 others, so that the $\frac{1}{2}(16 . 5)$ intersections are umbilics. Similarly, in the case of a cubic surface passing once through the imaginary circle, we have 27 right lines on the surface. One such line lies altogether at infinity, and is met by 10 other lines of the surface. Excluding these, we have 16 right lines of no length intersecting, as in the first case, and the $\frac{1}{2}(16 . 5)$ intersections are the umbilics. At an umbilic, the inflexional tangents are lines of no length, and therefore must, in the cases now considered, lie altogether on the surface.

II. *Solution by* Professor TOWNSEND, F.R.S.

For the case of the spherical cubic, to which that of the bispherical quartic may be reduced by inversion, this pretty property may be readily proved as follows:—

The complete intersection of the surface with the plane at infinity being the imaginary circle and a real right line, and the latter being intersected by ten (including all that are real) of the remaining twenty-six lines of the surface, consequently sixteen of those lines (all necessarily imaginary) pass through the circle, and, as the two inflexional tangents at every umbilic of the surface are two of the latter, the umbilical points are therefore grouped on those sixteen lines. Again, as each of those lines is intersected by five of themselves and by five real lines of the surface, and as every intersection of two of them is an umbilic of the surface, and is counted twice, the umbilici are therefore forty in number, and are grouped in fives on those sixteen lines. Finally, as the tangent plane at every umbilic intersects the surface in two of the sixteen and in one of the ten lines, the forty umbilical tangent planes pass by fours through the ten lines of the surface which intersect with its line at infinity.

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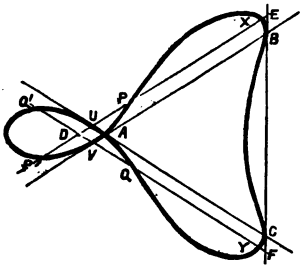
4042. (Proposed by Professor MANNHEIM.)—The tangents at a double point A on a quartic cut the curve again in B and C so that BC is a double tangent. Prove that $\frac{\rho_1 \rho_1'}{\rho_2 \rho_2'} = \frac{AB^3}{AC^3}$, where ρ_1, ρ_2 are the radii of curvature at A to the branches which touch AB, AC, and ρ_1', ρ_2' are the radii of curvature at B and C.

I. *Solution by* Professor TOWNSEND, F.R.S.

Through any point D infinitely near to the double point A, and within the loop of the curve determined by it, drawing parallels DE and DF to the two nodal tangents AB and AC, meeting the curve in the two tetrads of points PP'UX and QQ'VY and the double tangent BC at the two points E and F; then since, by Carnot's theorem,



$$\frac{DP.DP'.DU.DX}{EP.EP'.EU.EX} \cdot \frac{EB^2.EC^2}{FB^2.FC^2} \cdot \frac{FQ.FQ'.FV.FY}{DQ.DQ'.DV.DY} = 1,$$

$$\therefore \quad \frac{DP.DP'}{DV} \cdot \frac{EB^2}{EX} + \frac{DQ.DQ'}{DU} \cdot \frac{FC^2}{FY} = \frac{DY}{DX} \cdot \frac{EP.EP'.EU}{FQ.FQ'.FV} \cdot \frac{FB^2}{EC^2},$$

therefore in the limit $\frac{\rho_1 \sin BAC}{\rho_2 \sin BAC} \cdot \frac{\rho_1' \sin ABC}{\rho_2' \sin ACB} = \frac{AC}{AB} \cdot \frac{AB^3}{AC^3}$,

and therefore, &c.

II. *Solution by* Professor WOLSTENHOLME, M.A.

The equation of the quartic referred to the triangle ABC is of the form

$$y^2z^2 + x(\lambda y^3 + \mu y^2 z + \nu y z^2 + \rho z^3) + \sigma x^2 yz = 0.$$

At the point A, $x = 1$, $y = 0$, $z = 0$, and

limit of $\frac{y^2}{z}$ for the branch touching AB is $-\frac{\sigma}{\lambda}$,

" $\frac{z^2}{y}$ " " AC is $-\frac{\sigma}{\rho}$.

But the limit of $\frac{y^2}{z} = \frac{\rho_1}{R} \frac{\sin A \sin B}{\sin^2 C}$, and that of $\frac{z^2}{y} = \frac{\rho_2}{R} \frac{\sin A \sin C}{\sin^2 B}$.

At the point B, $y = 1$, $z = 0$, $x = 0$, and

the limit of $\frac{z^2}{x}$ is $-\frac{1}{\lambda} \equiv \frac{\rho_1'}{R} \frac{\sin B \sin C}{\sin^2 A}$,

at the point C, $z = 1$, $x = 0$, $y = 0$, and

the limit of $\frac{y^2}{x}$ is $-\frac{1}{\rho} \equiv \frac{\rho_2'}{R} \frac{\sin B \sin C}{\sin^2 A}$,

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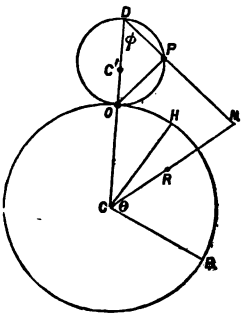
R being the radius of the circle ABC.

Hence $\frac{\lambda}{\rho} = \frac{\rho_2}{\rho_1}\frac{\sin^3 C}{\sin^3 B} = \frac{\rho_2'}{\rho_1'}$, or $\frac{\rho_1\rho_2'}{\rho_1'\rho_2} = \frac{\sin^3 C}{\sin^3 B} = \frac{AB^3}{AC^3}$.

4046. (Proposed by B. WILLIAMSON, M.A.)—Prove, geometrically, that the equation to the reciprocal polar of an epicycloid, with respect to the centre of the fixed circle, is of the form $r \sin m\theta = c$. [An *analytical* proof is given in Art. 311 of the new edition of Dr. SALMON's *Higher Curves*.]

I. *Solution by the* PROPOSER.



Let P be the position of the generating point at any instant. Take OH = arc OP, and make HB = 90°. Join CH, CB, CO, DP, and draw CN perpendicular to DP.

Let $\angle PDO = \phi$, $\angle BCN = \theta$, $CO = a$,

$OC' = b$, $CN = p$;

then $\theta = \frac{1}{2}\pi - HCN = HCO + PDO$

$= \phi\left(1 + \frac{2b}{a}\right)$;

therefore $\phi = m\theta$, where $m = \frac{a}{a+2b}$.

Again, $CN = CD \cdot \sin CDN$,

or $p = (a+2b)\sin m\theta$.

Now let R be the pole of DN with respect to the circle OHB, and since DN is a tangent to the epicycloid, the locus of R is the required reciprocal polar; consequently, if $CR = r$, we have

$$r \sin m\theta = \frac{a^2}{a+2b} = ma.$$

It may be observed that the reciprocal polar of the curve $r \sin m\theta = c$ is an hypo- or an epi- cycloid, according as m is $>$ or < 1.

II. *Solution by* Professor WOLSTENHOLME.

The radii of the fixed and moveable circles being a, b, if a circle be described of radius $a + 2b$, and with the fixed circle, and two points P, Q describe this circle with velocities in the ratio $a + b : b$, starting together from a vertex of the epicycloid (for an hypocycloid, we have only to put $-b$ instead of b), the joining line is in all positions a tangent to the epicycloid.

The equation of the tangent is then

$$x\cos\theta + y\sin\theta = (a+2b)\cos\frac{a}{a+2b}\theta,$$

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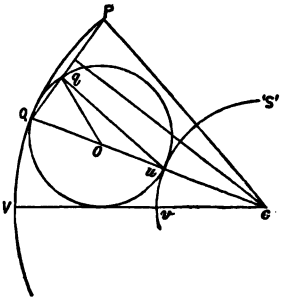
and the equation of the reciprocal polar is of course

$$\frac{k^2}{r} = (a+2b)\cos\frac{a}{a+2b}\theta.$$

If the prime radius pass through a cusp instead of a vertex, the equation is

$$\frac{k^2}{r} = (a+2b)\sin\frac{a}{a+2b}\theta.$$

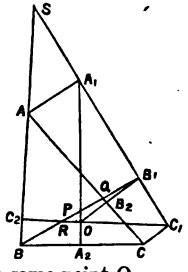


For if QP be a position of the moving line, C the centre of the fixed circle, which meets CQ in U, O the centre of the circle on UQ, whose radius is therefore b, $\angle VCQ = bz$; therefore $\angle QCP = az$, and QP meets the circle in QU in q, Uq is perpendicular to QP, and therefore $\angle QUq = \frac{1}{2}\angle QCP$, therefore $\angle QOq = az$, therefore arc Qq = arc abz = arc vU; therefore, if arc vS be equal to the semi-perimeter of the circle O, arc US = arc Uq, or q traces out an epicycloid to which QP is a tangent at q; QP is divided in a constant ratio $b : a+b$ in q. This method of proving properties of epicycloids and hypocycloids was communicated by me to the Mathematical Society a few weeks ago. Many of their properties are much more easily proved in this way; for instance, the property that the evolute is always a similar curve is almost obvious: producing qU to meet the fixed circle again in U', it is obvious that the angular velocity of U is equal to that of Q, and of U' to that of P, and therefore the property immediately follows, the ratio of their dimensions being $a : a+2b$.

3888. (Proposed by J. Collins.)—AA_1, BB_1, CC_1 are perpendiculars from the vertices A, B, C of a given triangle upon a straight line given in position, and A_1A_2, B_1B_2, C_1C_2 are perpendiculars upon the sides opposite the vertices A, B, C. Prove that the last three perpendiculars pass through the same point.

I. *Solution by* F. D. Thomson, M.A.

Let A_1A_2 meet C_1C_2 in Z, and meet B_1B_2 in Y. Then if S be the angle which $A_1B_1C_1$ makes with AB, we have from the geometry of the figure,



$$\frac{A_1Z}{A_1C_1} = \frac{\sin A_1C_1Z}{\sin A_1ZC_1} = \frac{\cos S}{\sin B};$$

$$\therefore \quad A_1Z = A_1C_1\frac{\cos S}{\sin B} = \frac{AC}{\sin B}\cos(A-S)\cos S.$$

Similarly $A_1Y = A_1B_1\dfrac{\cos(A-S)}{\sin C}$

$$= \frac{AB}{\sin C}\cos S\cos(A-S);$$

therefore $A_1Y = A_1Z$, or A_1A_2, B_1B_2, C_1C_2 meet in the same point O.

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[This theorem is a particular case of that which forms Ex. 7 of Art. 54 of SALMON'S *Conics* (5th edit.), the vertices of the second triangle being here three collinear points.]

II. *Solution by* Professor TOWNSEND, F.R.S.

For, since

$$\begin{aligned}&[(BA_2)^2-(CA_2)^2]+[(CB_2)^2-(AB_2)^2]+[(AC_2)^2-(BC_2)^2]\\&=[(BA_1)^2-(CA_1)^2]+[(CB_1)^2-(AB_1)^2]+[(AC_1)^2-(BC_1)^2]\\&=[(AC_1)^2-(AB_1)^2]+[(BA_1)^2-(BC_1)^2]+[(CB_1)^2-(CA_1)^2]\\&=[(A_1C_1)^2-(A_1B_1)^2]+[(B_1A_1)^2-(B_1C_1)^2]+[(C_1B_1)^2-(C_1A_1)^2]=0,\end{aligned}$$

therefore, &c.

III. *Solution by* N'IMPORTE.

Let O be the intersection of A_1A_2 with B_1B_2, and through C_1, O draw a line meeting AB at C_2. It is required to prove that C_1C_2 is perpendicular to AB.

Let P, Q, R be the intersections of BB_1 with A_1A_2, AC and C_1C_2; then the angle $BPA_2 = A_1PB_1$, and PA_2B, PB_1A_1 are right angles,

therefore the angle PBA_2 or $QBC = PA_1B_1$ or OA_1B_1 ..... (1).

Also $BQC = (QB_1B_2 \text{ or } PB_1O) + QB_2B_1 = PB_1O + PB_1A_1 = A_1B_1O$ ... (2),

consequently, from (1) and (2), the triangles BQC, A_1B_1O are similar...(3).

Again, the angle $AQB = B_2QB_1 = QB_2B_1 - (QB_1B_2 \text{ or } PB_1O)$

$= PB_1C - PB_1O = OB_1C_1$..... (4).

Also, $AQ : QC = A_1B_1 : B_1C_1$, and $QC : QB = B_1O : A_1B_1$, from (3),

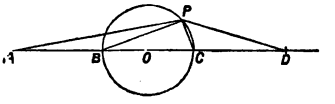
therefore $AQ : QB = B_1O : B_1C_1$ .... (5).

Now from (4) and (5) the triangles AQB and OB_1C_1 are equiangular, therefore the angle ABQ or $C_2BR = B_1C_1O$ or B_1C_1R, consequently the angle $RC_2B = RB_1C_1$, which is a right angle; that is, C_1C_2 is perpendicular to AB.

3998. (Proposed by H. MURPHY.)—If on the middle segment of a trisected line a circle be drawn, and if any point on its circumference be joined to the four points on the trisected line, prove that the product of the tangents of the extreme angles at this point is equal to $\frac{1}{4}$.

I. *Solution by* CHRISTINE LADD.

Let B, C be the points of trisection of the straight line AD; then, since BPC is a right angle, we have



$$\tan APB \tan CPD = \frac{\sin APB \sin CPD}{\sin APC \sin BPD}$$

$$= \frac{AB \cdot CD}{AC \cdot BD} = \frac{1}{4}.$$

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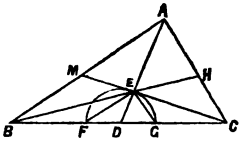
II. *Solution by the* PROPOSER.



The semicircle on the middle third FG of BC is evidently the locus of the intersection of the bisectors of the sides of the triangle ABC. Join the middle point D of FG to any point E on the circumference, and produce DE till DA = 3DE; join BA and CA, which are parallel to the sides of the triangle EFG; therefore $\angle ABE = BEF$ and $ACE = GEC$.

Now $\tan ABF = \tan BEF = \frac{HA}{AB}$, and $\tan GEC = \frac{MA}{CA}$;

therefore $\tan BEF \cdot \tan CEG = \frac{1}{4}$.

2021. (Proposed by N'IMPORTE.)—Find the sides of a right-angled triangle, such that its perimeter shall be a square, and the diameter of its inscribed circle a cube, or *vice versâ*, that is, such that the perimeter may be a cube, and the diameter of the inscribed circle a square.

Solution by S. BILLS; R. TUCKER, M.A.; *and others.*

Let $(p^2+q^2)x$, $(p^2-q^2)x$, and $2pqx$ denote the sides of the required right-angled triangle; then in the *first* part of the question we must have its perimeter $= 2p(p+q)x = \square = r^2$ (suppose), and the diameter of its inscribed circle $= 2q(p-q)x = \text{cube} = \frac{r^3}{s^3}$ (suppose),

therefore $x = \frac{r^2}{2p(p+q)} = \frac{r^3}{2q(p-q)s^3}$, whence $r = \frac{q(p-q)s^3}{p(p+q)}$.

As an example, take $p=2$, $q=1$, and $s=6$; then $r=36$ and $x=108$, and the sides of the triangle will be 540, 432, 324.

Again, for the *second* part of the question,

$$2p(p+q)x = \text{cube} = \frac{r^3}{s^3}, \quad 2q(p-q)x = \square = r^2;$$

therefore $x = \frac{r^2}{2q(p-q)} = \frac{r^3}{2p(p+q)s^3}$, whence $r = \frac{p(p+q)s^3}{q(p-q)}$.

Taking $p=2$, $q=1$, $s=1$, we have $r=6$, $x=18$; hence the sides of the triangle will be 90, 72, 54.

Many other answers may be readily found.

3810. (Proposed by G. S. CARR.)—Show that the law of central force for a particle describing the orbit $x^n + y^n = 1$ (n being an even integer) is

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$F \propto x^{n-2}y^{n-2}(x^2+y^2)^{\frac{1}{2}}$. When $n = \infty$ the orbit becomes a square, and the velocity at the corners is to that elsewhere as $1 : \sqrt{2}$.

Solution by the PROPOSER.

If F be the force to the centre, we know that

$$F = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) \text{ and } v^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} \ldots\ldots\ldots (1, 2),$$

TAIT and STEELE'S *Dynamics*, Arts. 127 and 133. We proceed to calculate F and v.

The equation $x^n + y^n = 1$ in polar coordinates becomes

$$u^n = \sin^n\theta + \cos^n\theta \text{ where } u = r^{-1} \ldots\ldots\ldots\ldots\ldots (3),$$

$$u^{n-1}\frac{du}{d\theta} = \sin^{n-1}\theta\cos\theta - \cos^{n-1}\theta\sin\theta,$$

$$(n-1)u^{n-2}\left(\frac{du}{d\theta}\right)^2 + u^{n-1}\frac{d^2u}{d\theta^2} = (n-1)\sin^{n-2}\cos^2\theta - \sin^n\theta + (n-1)\cos^{n-2}\sin^2\theta - \cos^n\theta$$

$$= (n-1)(\sin^{n-2}\theta + \cos^{n-2}\theta) - (n-1)(\sin^n\theta + \cos^n\theta) - (\sin^n\theta + \cos^n\theta)$$

$$= (n-1)(\sin^{n-2}\theta + \cos^{n-2}\theta) - nu^n \text{ by (3)};$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{1}{u^{n-1}}\left\{(n-1)(\sin^{n-2}\theta + \cos^{n-2}\theta) - nu^n - (n-1)u^{n-2}\left(\frac{du}{d\theta}\right)^2\right\};$$

$$\therefore u + \frac{d^2u}{d\theta^2} = \frac{n-1}{u^{n-1}}\left\{\sin^{n-2}\theta + \cos^{n-2}\theta - u^{n-2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]\right\} \ldots\ldots (4).$$

To find $u^2 + \left(\frac{du}{d\theta}\right)^2$, we have from above

$$\frac{du}{d\theta} = \frac{\sin^{n-1}\theta\cos\theta - \cos^{n-1}\theta\sin\theta}{u^{n-1}};$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{\sin^{2n-2}\theta\cos^2\theta + \cos^{2n-2}\theta\sin^2\theta - 2\sin^n\theta\cos^n\theta}{u^{2n-2}}$$

$$= \frac{1}{u^{2n-2}}\left\{\sin^{2n-2}\theta + \cos^{2n-2}\theta - \sin^{2n}\theta - \cos^{2n}\theta - 2\sin^n\theta\cos^n\theta\right\}$$

$$= \frac{1}{u^{2n-2}}\left\{\sin^{2n-2}\theta + \cos^{2n-2}\theta - u^{2n}\right\} \text{ by (3)}.$$

Therefore $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^{2n-2}}\left\{\sin^{2n-2}\theta + \cos^{2n-2}\theta\right\} \ldots\ldots\ldots\ldots (5).$$

Substitute this in (4); thus

$$u + \frac{d^2u}{d\theta^2} = \frac{n-1}{u^{n-1}}\left\{\sin^{n-2}\theta + \cos^{n-2}\theta - \frac{1}{u^n}(\sin^{2n-2}\theta + \cos^{2n-2}\theta)\right\}$$

$$= \frac{n-1}{u^{2n-1}}\left\{(\sin^{n-2}\theta + \cos^{n-2}\theta)(\sin^n\theta + \cos^n\theta) - (\sin^{2n-2}\theta + \cos^{2n-2}\theta)\right\}$$

$$= \frac{n-1}{u^{2n-1}}\sin^{n-2}\theta\cos^{n-2}\theta.$$

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Hence by (1) we have

$$F = h^2 \frac{n-1}{u^{2n-3}} \sin^{n-2}\theta \cos^{n-2}\theta = h^2(n-1)x^{n-2}y^{n-2}(x^2+y^2)^{\frac{1}{2}}$$

in rectangular coordinates.

Also the velocity in orbit by (2) and (5) is found from

$$v^2 = \frac{h^2}{u^{2n-2}}(\sin^{2n-2}\theta + \cos^{2n-2}\theta) = h^2(x^{2n-2}+y^{2n-2}) \quad \ldots\ldots (6).$$

Again $x^n + y^n = 1$ ..... (7),

and n is an even integer. Therefore, when n becomes infinite we must have, for all values of x less than 1, $y = \pm 1$, and for all values of y less than 1, $x = \pm 1$. Hence the curve becomes a square in the limit.

Also, if p be the perpendicular upon the tangent, we have $vp = h$. But while the particle is describing the sides of the square, $p = 1$ always; and at the corners p evidently must pass through the value $\sqrt{2}$ = half a diagonal. Hence the velocity at the corners is to that elsewhere as $1 : \sqrt{2}$. That $p = \sqrt{2}$ at the corners may be proved by putting $x = y$ in equations (6) and (7). This gives

$$h = (2)^{\frac{1}{n}-\frac{1}{2}} \text{ which becomes } v = \tfrac{1}{2}h\sqrt{2} \text{ when } n = \infty.$$

When the orbit becomes a square, the law of force is that $F = 0$, excepting at the angular points; and when the particle arrives at those points, F suddenly takes the value $h^2(n-1)\sqrt{2} = \infty$ when n is infinite. That is, an *infinite* but *momentary* impulsive force changes the direction of motion *instantaneously*.

2020. (Proposed by N'IMPORTE.)—If a curve of the third order have a double point A, and be cut by any straight line in B, C, D; and if, when ABC is taken as triangle of reference, the tangents at A are represented by the equation $P\beta^2 + Q\beta\gamma + R\gamma^2 = 0$, and the tangents at B, C by the equations $P\alpha + N\gamma = 0$, and $M\beta + R\alpha = 0$; show that the equation to the straight line AD is $N\beta + M\gamma = 0$, and find the equation of the curve.

Solution by Professor CLIFFORD.

Since $P\beta^2 + Q\beta\gamma + R\gamma^2 = 0$ and $P\alpha + N\gamma = 0$ are the tangents at the points where $\gamma = 0$ meets the curve, its equation must be of the form

$$(P\beta^2 + Q\beta\gamma + R\gamma^2)(P\alpha + N\gamma) = \gamma^2\phi;$$

also of the form $(P\beta^2 + Q\beta\gamma + R\gamma^2)(M\beta + R\alpha) = \beta^2\chi$.

It will be found that the equation

$$\alpha(P\beta^2 + Q\beta\gamma + R\gamma^2) + \beta\gamma(N\beta + M\gamma) = 0 \quad \ldots\ldots (1)$$

is of both these forms. For clearly the lines $P\beta^2 + Q\beta\gamma + R\gamma^2 = 0$ only meet the curve at the point $(\beta\gamma)$, and the lines $P\alpha + N\gamma = 0$, $M\beta + R\alpha = 0$, *touch* the curve at the points $(\alpha\gamma)$, $(\alpha\beta)$. It is now obvious that the other point where α meets the curve is in the line $N\beta + M\gamma = 0$, which is therefore the line AD.

We may notice that the tangents drawn from D to the curve are represented by $\{Q \pm 2(PR)\}\alpha + N\beta + M\gamma = 0$ ..... (2);

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and there are only two because there is a double point. The lines drawn from A to their points of contact are represented by $\beta\sqrt{P} \pm \gamma\sqrt{R} = 0$; hence these form an *harmonic pencil* with AB, AC.

The equation to the tangent at D is

$$N^2M\beta + M^2N\gamma = (M^2P + N^2R - MNQ)\alpha \quad \ldots\ldots\ldots\ldots (3),$$

and that to the line joining A with the other point where (3) meets the curve is $MP\beta + NR\gamma = 0$; hence the condition that D may be a *point of inflexion* is $PM^2 = RN^2$.

NOTE ON SPHERICAL TRIGONOMETRY. *By* S. A. RENSHAW.

The fundamental formulæ of Spherical Trigonometry may all be easily constructed (by which I mean composed or put together, not geometrically, but equationally) and remembered by the following method:—

Call the three sides α, β, γ and the angles opposite to them A, B, C, then in every case of the solution of Spherical Triangles there are three parts given, *one of which must be a side*, to find a fourth part.

The connecting equations are as follows:—

Equation (1) Between 3 sides, and one angle.

,, (2) ,, 2 sides and two opposite angles.

,, (3) ,, 2 sides and their included angle and the angle opposite one of them.

,, (4) ,, 1 side and 3 angles.

Let α_1, α_2, α_3, α_4 signify $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$, &c., &c.

Then Equation (1) is $\left.\begin{array}{ll} \alpha_2 = \beta_2\gamma_2 + \beta_1\gamma_1A_2 & \text{from } \alpha\beta\gamma A \\ \text{or } \beta_2 = \alpha_2\gamma_2 + \alpha_1\gamma_1B_2 & ,, \quad \beta\alpha\gamma B \\ \text{or } \gamma_2 = \alpha_2\beta_2 + \alpha_1\beta_1C_2 & ,, \quad \gamma\alpha\beta C \end{array}\right\}$;

,, (2) is $\dfrac{A_1}{\alpha_1} = \dfrac{B_1}{\beta_1} = \dfrac{C_1}{\gamma_1}$;

,, (3) is $\left.\begin{array}{ll} \alpha_4\beta_1 = \beta_2C_2 + C_1A_4 & \text{from } \alpha\beta CA \\ \alpha_4\gamma_1 = \gamma_2B_2 + B_1A_4 & ,, \quad \alpha\gamma BA \\ \text{\&c.} \quad \text{\&c.} & \end{array}\right\}$;

,, (4) is $\left.\begin{array}{l} A_2 = -B_2C_2 + B_1C_1\alpha_2 \text{ from } ABC\alpha \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\}$.

If, then, three sides and an angle be the four parts between which the existing equation is required, as in equation (1), write the combination with that side first which is opposite to the given angle; *e.g.*, suppose $\alpha\beta\gamma$B be given, write it $\beta\alpha\gamma$B, then form the expression $\beta = \alpha\gamma + \alpha\gamma$B, and add the subscripts always in the same order, viz., 2, 22, 112; and obtain $\beta_2 = \alpha_2\gamma_2 + \alpha_1\gamma_1B_2$, the equation required. Equation (4) is constructed in exactly the same manner as (2), only that the sign of the first number on the right hand is changed to (−).

Equation (2) is too simple to need any explanation.

Equation (4). Write that side first opposite the given angle, and the angle itself last, and the other two parts between them. Suppose αBAC be

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given; write it αBCA, and form the expression $\alpha\beta = \beta C + CA$ (the *regular* method of constructing which is evident); and add the subscripts 41, 22, 14 always in the same order. We thus obtain $\alpha_4\beta_1 = \beta_2 C_2 + C_1 A_4$.

The following table shows the parts omitted, the parts given, the order in which the combination in each case is written, the subscripts for each case, and the resulting equations:—

Parts omitted.	Parts given.	Order of writing Combination.	Subscripts.	Resulting Equations.	
$\alpha\beta$	γABC	CABγ	2, 22, 112	$C_2 = -A_2B_2 + A_1B_1\gamma_2$	
$\alpha\gamma$	βABC	BCAβ	,,	$B_2 = -C_2A_2 + C_1A_1\beta_2$	... (1).
$\beta\gamma$	αABC	ABCα	,,	$A_2 = -B_2C_2 + B_1C_1\alpha_2$	
αA	$\beta\gamma$BC	$\beta\gamma$BC	1, 1, 1, 1	$\beta_1C_1 = \gamma_1B_1$	
βB	$\alpha\gamma$AC	$\alpha\gamma$AC	,,	$\alpha_1C_1 = \gamma_1A_1$	..... (2).
γC	$\alpha\beta$AB	$\alpha\beta$AB	,,	$\alpha_1\beta_1 = \beta_1A_1$	
αB	$\beta\gamma$AC	$\gamma\beta$AC	41, 22, 14	$\gamma_4\beta_1 = \beta_2A_2 + A_1C_4$	
αC	$\beta\gamma$AB	$\beta\gamma$AB	,,	$\beta_4\gamma_1 = \gamma_2A_2 + A_2B_4$	
βA	$\alpha\gamma$BC	$\gamma\alpha$BC	,,	$\gamma_4\alpha_1 = \alpha_2B_2 + B_1C_4$	..... (3).
βC	$\alpha\gamma$BA	$\alpha\gamma$BA	,,	$\alpha_4\gamma_1 = \gamma_2B_2 + B_1A_4$	
γA	$\alpha\beta$CB	$\beta\alpha$CB	,,	$\beta_4\alpha_1 = \alpha_2C_2 + C_1B_4$	
γB	$\alpha\beta$CA	$\alpha\beta$CA	,,	$\alpha_4\beta_1 = \beta_2C_2 + C_1A_4$	
AB	$\alpha\beta\gamma$C	$\gamma\alpha\beta$C	2, 22, 112	$\gamma_2 = \alpha_2\beta_2 + \alpha_1\beta_1C_2$	
AC	$\alpha\beta\gamma$B	$\beta\gamma\alpha$B	,,	$\beta_2 = \gamma_2\alpha_2 + \gamma_1\alpha_1B_2$	..... (4).
BC	$\alpha\beta\gamma$A	$\alpha\beta\gamma$A	,,	$\alpha_2 = \beta_2\gamma_2 + \beta_1\gamma_1A_2$	

Now in order to apply the above to the solution of right-angled triangles, or a case in which one of the parts = 90 degrees, we have only to make the part in the above = 1 if it have the subscript 1, and if it have the subscript 2 or 4 to make the term in which it is involved vanish. Thus suppose A is right, and β, B are given, to find γ; then by the combination $\beta\gamma$AB in (3), remembering the term involving A_2 vanishes, we get $\beta_4\gamma_1 = B_4$, or what is the same thing, $\gamma_1 = \beta_2 B_4$. Again, suppose A right, and $\alpha\gamma$ given, to find B, take the combination $\alpha\gamma$BA in (3), and we get $\alpha_4\gamma_1 = \gamma_2 B_2$ or $B_2 = \alpha_4\gamma_3$. Or, again, suppose A right, and βB given, to find C; take the combination BCAβ in (1), and derive $B_2 = C_1\beta_2$ or $C_1 = \dfrac{B_2}{\beta_1}$ as required.

It will be observed that in the combinations (CABγ, &c.) the angle and side opposed to one another are always written first and last, and that the other two parts fall in naturally between them, in the equations (1), (3), and (4). The method of forming combinations and equations (2) is evident.

1920. (Proposed by the EDITOR.)—If in the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-2}x^2 + p_{n-1}x + p_n = 0,$$

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we have e^{π}, or $e^{\frac{1}{2}\pi}$, or $e^{\frac{1}{3}\pi}$ as a root; show that, in the three cases respectively, we shall have the relations

$$p_0+p_2+p_4+\&c. = p_1+p_3+p_5+\&c. \quad \ldots\ldots (1),$$

$$\left.\begin{aligned} p_1+p_5+p_9+\&c. &= p_3+p_7+p_{11}+\&c. \\ p_0+p_4+p_8+\&c. &= p_2+p_6+p_{10}+\&c. \end{aligned}\right\} \quad \ldots\ldots (2),$$

$$\left.\begin{aligned} p_1+p_2+p_7+p_8+\&c. &= p_4+p_5+p_{10}+p_{11}+\&c. \\ p_1-p_2-p_4+p_5+\&c. &= 2(p_0-p_3+p_6-p_9+\&c.) \end{aligned}\right\} \quad \ldots\ldots (3).$$

Solution by N'Importe.

Take the *general* case of $e^{i\alpha}$ as a root, then $e^{-i\alpha}$ will *also* be a root; hence, dividing all through by x^n, and substituting for x the values $e^{i\alpha}$, $e^{-i\alpha}$, we have

$$0 = p_0+p_1e^{-i\alpha}+p_2e^{-2i\alpha}+p_3e^{-3i\alpha}+\ldots\ldots (1),$$

$$0 = p_0+p_1e^{i\alpha}+p_2e^{2i\alpha}+p_3e^{3i\alpha}+\&c. \quad \ldots\ldots (2).$$

From (2) − (1) and (2) + (1) we get

$$0 = p_1\sin\alpha+p_2\sin 2\alpha+p_3\sin 3\alpha+\&c. \quad \ldots\ldots (3),$$

$$0 = p_0+p_1\cos\alpha+p_2\cos 2\alpha+p_0\cos 3\alpha+\&c. \quad \ldots\ldots (4).$$

Now give α the values π, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, and we obtain, from (3) and (4), in the three cases respectively, the relations given in the question.

3959. (Proposed by J. J. Sylvester, F.R.S.)—Prove (1) that the equation $x^4+4ex^3+6e^2x^2+4ex+1=0$ has no real roots if $\pm e$ lies without the limits $\frac{1}{2}$ and 1, and has two real and two imaginary roots if $\pm e$ lies within these limits. Prove also (2) that the equation

$$x^5+5ex^4+10e^2x^3+10e^3x^2+5ex+1=0$$

has only one real root (viz., $x=-1$) if e lies without the limits $\frac{1}{2}$ and 1, and has two real and two imaginary roots if e lies within these limits.

Solution by Professor Wolstenholme, M.A.

1. This equation gives

$$(x+x^{-1})^2+4e(x+x^{-1})+4e^2 = 2(1-e^2),$$

whence we obtain $x+x^{-1} = -2e \pm \left\{2(1-e^2)\right\}^{\frac{1}{2}}$;

but for real values of x, $x+x^{-1}$ cannot lie between 2 and -2; hence, supposing e positive,

$$2e+\left\{2(1-e^2)\right\}^{\frac{1}{2}} > 2, \quad 1-e^2 > 2(1-e)^2, \quad (1-e)(3e-1) > 0,$$

or e lies between $\frac{1}{3}$ and 1.

Since in this case $2e-\left\{2(1-e^2)\right\}^{\frac{1}{2}}$ is less than 2, two of the roots will be real and two impossible. Hence the equation will have no real roots unless $e^2 < 1$ and $> \frac{1}{3}$; and if e^2 lie between these limits, two roots will be real and two impossible.

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2. From this equation, when the factor $x+1$ is got rid of, we get

$$x+x^{-1} = \tfrac{1}{2}\left[1-5e \pm \left\{5(1-e)(1+3e)\right\}^{\frac{1}{2}}\right];$$

and for real values of $x+x^{-1}$, e cannot be > 1. For real values of x,

$$(x+x^{-1})^2 > 4,$$

or $$(1-5e)^2 + 5(1-e)(1+3e) \pm 2(1-5e)\left\{5(1-e)(1+3e)\right\}^{\frac{1}{2}} > 16,$$

or $$\pm (1-5e)\left\{5(1-e)(1+3e)\right\}^{\frac{1}{2}} > 5(1-e^2),$$

$$(1-5e)^2(1+3e) > 5(1-e)(1+e)^2;$$

or $0 > 4(1+3e-20e^3)$, or $0 > (1-2e)(1+5e+10e^2)$, or $e > \frac{1}{2}$.

Hence if e lie between $\frac{1}{2}$ and 1, the equation will have one pair of real roots, and obviously one pair of impossible roots.

2015. (Proposed by T. T. WILKINSON, F.R.A.S.)—If the sides of any octagon inscribed in a conic be produced to meet, prove that their intersections will lie on another conic.

Solution by the PROPOSER.

This property is an immediate consequence of some elegant theorems in SALMON'S *Higher Plane Curves*. It is there shown that all curves of the nth degree which pass through $\frac{1}{2}n(n+3)-1$ fixed points pass also through $\frac{1}{2}(n-1)(n-2)$ other fixed points. From this it immediately follows that if of the n^2 points of intersection of *two* curves of the nth degree, np lie on a curve of the pth degree, $(p<n)$, the remaining $n(n-p)$ points will lie on a curve of the $(n-p)$th degree. In these investigations it is not necessary that the curves be proper ones, since the properties hold good when the functions break up into factors and represent right lines. Hence finally:—If a polygon of $2n$ sides be inscribed in a conic, the $n(n-2)$ points where each odd side intersects the non-adjacent even sides will lie in a curve of the $(n-2)$nd degree. In the example given in the question $n=4$, and consequently the intersections lie on a curve of the degree $(n-2)$, or the second degree, and is consequently a conic.

3884. (Proposed by W. GODWARD.)—A point C is taken in the diameter AB of a semicircle, and on AC, BC other semicircles are described; also a series of circles whose centres are P_1, P_2, P_3 ..., are constructed such that each one has contact with the preceding and the semicircles on AB, AC; the first circle P_1 having contact with the semicircle on BC. Prove that the loci of P_1, P_2, P_3 ... are ellipses having a common focus.

Solution by the PROPOSER.

Let O, the middle point of AB, be the origin of coordinates, x and y the

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coordinates of P_n, the centre of the nth circle; and r_n and R the radii of (P_n) and (O). Then by a well known property of the *arbelon* (LESLIE'S *Geometrical Analysis*, p. 115) we have $y = 2nr_n$, or $r_n = \frac{y}{2n}$. It will be at once seen that

$$x^2 + y^2 = (R - r_n)^2 = \left(R - \frac{y}{2n}\right)^2,$$

which reduces to

$$\left(1 - \frac{1}{4n^2}\right) y^2 + x^2 + \frac{R}{n} y - R^2 = 0,$$

and this is the equation to an ellipse, the coordinates of the centre being 0 and $-\frac{2nR}{4n^2-1}$, and semi-axes $a = \frac{4n^2R}{4n^2-1}$, and $b = \frac{2nR}{(4n^2-1)^{\frac{1}{2}}}$.

By making $n = 1, 2, 3 \ldots$, we shall have the equations, coordinates of the centres, and semi-axes of the several ellipses.

Also the latus rectum $= \frac{2b^2}{a} = 2r$; and as this is independent of n, it is manifest that AB and O are the latus rectum and *common* focus of all the ellipses.

NOTE.—I proposed the case $n = 1$ as a question in the *Gentleman's Diary* for 1836, to which a very complete and elegant geometrical solution was given by my late lamented friend Dr. RUTHERFORD in the *Diary* for 1837. The question, for this simple case, has since found its way into some Cambridge books, and was given as a Senate House problem in 1871.

2028. (Proposed by GEOMETRICUS.)—a, b, c, d étant les quatre côtés successifs d'un quadrilatère convexe circonscrit à un cercle de rayon r, A l'angle de ce polygone compris par les côtés a et d, et δ la distance du sommet de cet angle au point de contact du côté a, démontrer qu'on a la relation

$$ad\,(2\delta - a + b)\cos^2 \tfrac{1}{2}A = (a + c)\,\delta^2.$$

Solution by the EDITOR.

Let C be the angle opposite to A; then the length of each of the tangents from C is $(\delta - a + b)$, or $(\delta + c - d)$; therefore $c - d = b - a$, and $a - d = b - c$. Also the square of the diagonal opposite to A is

$$(a-d)^2 + 4ad \sin^2 \tfrac{1}{2}A, \text{ or } (b-c)^2 + 4bc \sin^2 \tfrac{1}{2}C;$$

therefore
$$ad \sin^2 \tfrac{1}{2}A = bc \sin^2 \tfrac{1}{2}C \quad \ldots\ldots (1).$$

And
$$r = \delta \tan \tfrac{1}{2}A = (\delta - a + b) \tan \tfrac{1}{2}C \quad \ldots\ldots (2).$$

$$\frac{(1)}{(2)^2} \text{ gives } \frac{ad}{\delta^2}(\delta - a + b)^2 \cos^2 \tfrac{1}{2}A = bc \cos^2 \tfrac{1}{2}C \quad \ldots\ldots (3).$$

$$(1) + (3) \text{ gives } \frac{ad}{\delta^2}(2\delta - a + b)\cos^2 \tfrac{1}{2}A = \frac{bc - ad}{b - a}.$$

But
$$\frac{bc - ad}{b - a} = c + \frac{ac - ad}{c - d} = a + c;$$

therefore
$$ad\,(2\delta - a + b)\cos^2 \tfrac{1}{2}A = (a + c)\,\delta^2.$$

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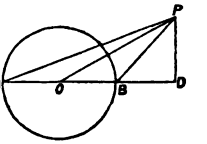
1330. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—On AB, the diameter of a circle, whose radius AO = OB = unity, let PD be a perpendicular drawn from any point P within or without the circle; then prove that $\cos^{-1}(OD + i \cdot PD) = \cos^{-1}\frac{1}{2}(PA - PB) + \cos^{-1}\frac{1}{2}(PA + PB)$.

Solution by the Proposer; R. Tucker, M.A.; S. Watson; *and others.*



Denoting the right-hand member of the equation by $\alpha + \beta$, we have

$\cos\alpha = \frac{1}{2}(PA - PB)$, $\cos\beta = \frac{1}{2}(PA + PB)$;

$\therefore \cos\alpha\cos\beta = \frac{1}{4}(PA^2 - PB^2) = \frac{1}{4}(AD^2 - BD^2)$

$= \frac{1}{4}(AD - BD)(AD + BD)$

$= AO \cdot OD = OD$;

and $\sin^2\alpha\sin^2\beta = (1 - \cos^2\alpha)(1 - \cos^2\beta)$

$= 1 - (\cos^2\alpha + \cos^2\beta) + (\cos\alpha\cos\beta)^2 = 1 - \frac{1}{2}(PA^2 + PB^2) + OD^2$

$= 1 - (AO^2 + OP^2) + OD^2 = -OP^2 + OD^2 = -PD^2$;

therefore $\sin\alpha\sin\beta = \pm i \cdot PD$;

and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = OD + i \cdot PD$,

or $\alpha + \beta = \cos^{-1}(OD + i \cdot PD)$.

[This theorem determines geometrically the rational and irrational parts of an imaginary angle when the rational and irrational parts of the cosine are known, and *vice versâ*.]

4075. (Proposed by David Trowbridge.)—Prove that

$$\frac{1}{1+nr} = 1 - \frac{nr}{1+r} + \frac{n(n-1)r^2}{(1+r)(1+2r)} - \frac{n(n-1)(n-2)r^3}{(1+r)(1+2r)(1+3r)} + \ldots\ldots$$

Solution by Rev. J. L. Kitchin, M.A.

Assume $\frac{1}{1+nr} = 1 + A_1 n + A_2 n(n-1) + A_3 n(n-1)(n-2) + \&c.$,

and this is legitimate for all values of n.

First, let $n = 1$, then $1 + A_1 = \frac{1}{1+r}$, therefore $A_1 = -\frac{r}{1+r}$.

Put now $n = 2$, and we get

$$\frac{1}{1+2r} = 1 - \frac{2r}{1+r} + 2A_2, \quad \text{whence } A_2 = \frac{r^2}{(1+r)(1+2r)}.$$

Similarly, putting $n = 3$, and substituting A_1, A_2, we get

$$A_3 = -\frac{r^3}{(1+r)(1+2r)(1+3r)}, \ \&c.;$$

$$= 1 - \frac{nr}{1+r} + \frac{n(n-1)r^2}{(1+r)(1+2r)} - \frac{n(n-1)(n-2)r^3}{(1+r)(1+2r)(1+3r)} + \&c.$$

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2216. (Proposed by the EDITOR.)—From any point on an ellipse or hyperbola draw two tangents to the circle on the minor axis as diameter; then, if from the ends of the diameter conjugate to that passing through the given point parallels be drawn to the tangents, prove that they will pass through the foci.

Solution by N'IMPORTE.

The equations of the tangents from a point (x_1, y_1) to the circle $x^2 + y^2 = b^2$ are $b(xx_1 + yy_1 - x_1^2 - y_1^2) \pm (xy_1 - x_1y)(x_1^2 + y_1^2 - b^2)^{\frac{1}{2}} = 0$ ..... (1).

If (x_1, y_1) is on the ellipse $a^2y_1^2 + b^2x_1^2 = a^2b^2$, then (1) takes the form

$$y = \frac{b^2x_1x}{a^2(be - y_1)} \pm \frac{b(b^2 + e^2x_1^2)}{ex_1^2 - by_1} \quad \ldots\ldots\ldots\ldots (2).$$

The coordinates of the ends of the diameter conjugate to that through (x_1, y_1) are $\left(\pm\frac{ay_1}{b}, \mp\frac{bx_1}{a}\right)$, and the equations of parallels to (2) through these points are $a^2(be - y_1)\,y = b^2x_1(x \pm ae)$ ..... (3), which pass through the foci $(0, \pm ae)$.

4045. (Proposed by J. J. WALKER, M.A.)—Find the equations which determine the position of the tangent plane to an ellipsoid such that the triangle whose three corners are the points in which it meets the axes may have the minimum area; and consider in particular the cases of the sphere and the prolate spheroid having the square of each minor axis equal to half the square of the major.

Solution by the Rev. J. L. KITCHIN, M.A.

The equation to the tangent plane of a central ellipsoid is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

where x', y', z' are the coordinates of the point of contact, and are connected by the relation $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1$ ..... (1).

Now coordinates of points where the plane meets the axes x, y, z, are

$$\left(\frac{a^2}{x'}, 0, 0\right), \quad \left(0, \frac{b^2}{y'}, 0\right), \quad \left(0, 0, \frac{c^2}{z'}\right),$$

therefore the areas on the planes xy, yz, xz, are

$$\frac{a^2b^2}{x'y'}, \quad \frac{b^2c^2}{y'z'}, \quad \frac{a^2c^2}{x'z'}.$$

therefore (area of triangle in question)$^2 = \frac{a^4b^4}{x'^2y'^2} + \frac{b^4c^4}{y'^2z'^2} + \frac{a^4c^4}{z'^2x'^2}$ ... (2),

which is to be a minimum subject to (1).

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Differentiating (1) and (2), we get, dropping the accents,

$$\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0,$$

and $$0 = \frac{a^4}{x^3}\left(\frac{c^4}{z^3} + \frac{b^4}{y^3}\right)dx + \frac{b^4}{y^3}\left(\frac{c^4}{z^2} + \frac{a^4}{x^2}\right)dy + \frac{c^4}{z^3}\left(\frac{b^4}{y^2} + \frac{a^4}{x^2}\right)dz.$$

Multiply the first by λ, and subtract, and put coefficients of dx &c. $= 0$, and we get

$$\lambda\frac{x}{a^2} = \frac{a^4}{x^3}\left(\frac{c^4}{z^3}+\frac{b^4}{y^3}\right), \quad \lambda\frac{y}{b^2} = \frac{b^4}{y^3}\left(\frac{c^4}{z^2}+\frac{a^4}{x^2}\right), \quad \lambda\frac{z}{c^2} = \frac{c^4}{z^3}\left(\frac{b^4}{y^2}+\frac{a^4}{x^2}\right)\ldots(3),$$

therefore

$$\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = \frac{a^4}{x^2}\left(\frac{c^4}{z^2} + \frac{b^4}{y^2}\right) + \frac{b^4}{y^2}\left(\frac{c^4}{z^2} + \frac{a^4}{x^2}\right) + \frac{c^4}{z^2}\left(\frac{b^4}{y^2} + \frac{a^2}{x^2}\right),$$

therefore $$\lambda = \frac{2a^4c^4}{x^2z^2} + \frac{2b^4a^4}{x^2y^2} + \frac{2c^4b^4}{y^2z^2}.$$

Whence the equations (3) become

$$\frac{2x^2}{a^2}\left(\frac{a^4c^4}{x^2z^2} + \frac{a^4b^4}{x^2y^2} + \frac{b^4c^4}{y^2z^2}\right) = \frac{a^4}{x^2}\left(\frac{c^4}{z^2} + \frac{b^4}{y^2}\right), \text{ &c.}$$

From these three equations we get the point of contact of the tangent plane.

1. In the sphere, we have $a = b = c$,

therefore $$\frac{2x^2}{a^2}\left(\frac{a^8}{x^2z^2} + \frac{a^8}{x^2y^2} + \frac{a^8}{y^2z^2}\right) = \frac{a^8}{x^2z^2} + \frac{a^8}{y^2x^2},$$

therefore $$\frac{2x^2}{a^2}\cdot\frac{a^2}{x^2y^2z^2} = \frac{y^2+z^2}{x^2y^2z^2},$$

therefore $2x^2 = y^2+z^2 = a^2-x^2$, therefore $x = y = z = \pm\frac{1}{3}a\sqrt{3}$,

which gives eight points of contact of the minimum plane.

The points of contact are thus determined; in the sphere inscribe a cube, and through the angular points draw tangent planes; any one of these is the minimum plane.

2. In the case of the prolate spheroid, $a^2 = 2b^2 = 2c^2$;

therefore $$\frac{2x^2}{a^2}\left(\frac{a^8}{4x^2z^2} + \frac{a^8}{4x^2y^2} + \frac{a^8}{16y^2z^2}\right) = \frac{a^4}{4x^2}\left(\frac{a^4}{z^2} + \frac{a^4}{y^2}\right);$$

$$\frac{2x^2}{a^2}\left(\frac{4y^2+4z^2+x^2}{4x^2y^2z^2}\right) = \frac{y^2+z^2}{x^2y^2z^2}, \text{ therefore } \frac{2x^2}{a^2}(4y^2+4z^2+x^2) = 4(y^2+z^2).$$

But $x^2+2y^2+2z^2 = a^2$, therefore $x^2(2a^2-2x^2+a^2) = a^2(a^2-x^2)$,

whence we obtain $$x = \pm\tfrac{1}{3}(\sqrt{5}\pm1)\,a.$$

We may find y, z, and thus determine the point required. It is easy to see what the geometrical construction for it is.

1336. (Proposed by R. Tucker, M.A.)—A perfectly elastic ball is placed at one focus of an elliptic billiard table; show that the ball, however struck, will ultimately—everything being supposed perfectly smooth—move along the major axis.

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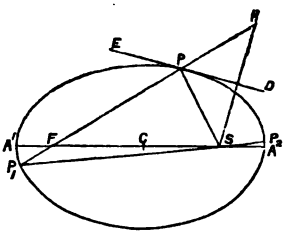
Solution by ALBERT ESCOTT, M.A.

Let S be the focus from which the ball is struck, and the first point of contact. Then, since the angle SPD = FPE, the ball will pass through the focus F, impinge upon the ellipse at P_1 and proceed through S to P_2, &c. Let the angle $ASP = 2\theta$, $AFP = A'FP' = 2\theta_1$, &c. Produce FP, and make PH = PS, or FH = AA'. Then, since



$$\frac{FH+FS}{FH-FS} = \frac{\tan\frac{1}{2}(FSH+FHS)}{\tan\frac{1}{2}(FSH-FHS)},$$

or $$\frac{1+e}{1-e} = \frac{\tan\frac{1}{2}ASP}{\tan\frac{1}{2}AFP} = \frac{\tan\theta}{\tan\theta_1};$$ therefore $\tan\theta_1 = \left(\frac{1-e}{1+e}\right)\tan\theta$.

Similarly $$\tan\theta_2 = \left(\frac{1-e}{1+e}\right)\tan\theta_1 = \left(\frac{1-e}{1+e}\right)^2\tan\theta,$$

and $$\tan\theta_n = \left(\frac{1-e}{1+e}\right)^n\tan\theta.$$

Let n be infinite; then the right-hand side of the equation vanishes, whatever the value of θ, and the ball moves along the major axis.

[A perfectly elastic ball, in impinging upon any surface, would be reflected so as to make the angle of reflection equal to the angle of incidence. If therefore the ball, *wherever placed*, be struck through one focus of the smooth elliptical billiard table, it will, after reflection at the edge of the table, pass through the other focus, since the normal at any point of an ellipse bisects the angle between the focal radii of that point. After each reflection the ball will continue to pass alternately through each focus, its line of direction making each time a smaller angle with the major axis; and thus its ultimate position, after an infinite number of reflections, would be the major axis itself.]

3806. (Proposed by Professor TOWNSEND, F.R.S.)—Defining, as the harmonic cone of intersection of two quadrics with respect to a point, the cone of the second order generated by the system of lines through the point, which are cut harmonically by the quadrics; show that the harmonic cone of intersection of any two quadrics U and V, with respect to any point, α', β', γ', δ', passes through the four lines of intersection of the two cones which, from the point as vertex, envelope any two quadrics $U+kV$ and $U-kV$ concurrent with and harmonically conjugate to each other with respect to U and V.

Solution by the PROPOSER.

The equations of the three cones in question (see Salmon's *Conic Sections*, 5th ed., p. 295, Note) being respectively,

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$$UV' + U'V - 2PQ = 0,$$
$$(U + kV)(U' + kV') - (P + kQ)^2 = 0,$$
$$(U - kV)(U' - kV') - (P - kQ)^2 = 0,$$

where P and Q are the polar planes of $\alpha', \beta', \gamma', \delta'$ with respect to U and V, and the two latter giving at once the first by subtraction, therefore, &c.

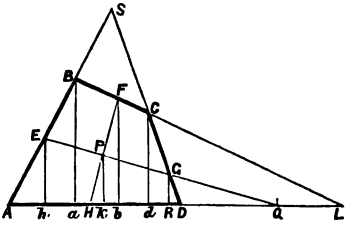
2218. (Proposed by the EDITOR.)—Find the point at which a triangle must be fixed; so that when it is struck at an angle by a blow perpendicular to its plane, it may begin to revolve about an axis parallel to the opposite side.

Solution by STEPHEN WATSON.

In any triangle the line from one of the angles to the middle of the opposite side passes through its centre of gravity; hence, if the triangle be fixed at any point in this line, and then struck at the angle by a blow perpendicular to its plane, it will begin to revolve about a line parallel to the side bisected by the line.

2029. (Proposed by the EDITOR.)—To divide a given quadrilateral into four equal parts, by two straight lines intersecting each other at right angles.

I. *Solution by* N'IMPORTE.



Let ABCD be the given quadrilateral, and EG, FH the lines required, intersecting perpendicularly at P, and dividing the figure into four equal parts. Produce AD, BC to meet in L, and AB, DC to meet in S; then the triangles DCL and BCS will be given in magnitude, and if half the quadrilateral be added to each, the triangles HFL and EGS will be given in magnitude; therefore because the angle S is given, the rectangle ES.SG will be given. Let perpendiculars be drawn upon AL from the points E, B, F, C, G, and produce EG to meet AL in Q.

Put $a^2 = 2\triangle HFL$, $b^2 = ES.SG$, $A^2 =$ area of quadrilateral ABCD, $c = AL$, $d = AD$, $e = AB$, $h = CD$, $m = Ba$, $n = Aa$, $k = Cd$, $p = aL$, $r = AS$, $s = DS$, $x = LH$, $y = AQ$, $z = ES$.

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Then $SG = \frac{b^2}{z}$, and $GD = s - \frac{b^2}{z} = \frac{sz-b^2}{z}$; but

$$CD : CD = GD : GR = \frac{k}{h}\left(\frac{sz-b^2}{z}\right);$$

therefore $2\triangle DGQ = \frac{k}{h}\left(\frac{sz-b^2}{z}\right) DQ = (y-d)(sz-b^2)\frac{k}{hz}$,

to which if A^2, or 2AEGD, be added, we have

$$2AEQ = (y-d)(sz-b^2)\frac{k}{hz} + A^2;$$

which is also $= AQ \cdot Eh$. But $BA : Ba = AE : Eh = \frac{m}{e}(r-z)$, therefore

$$\frac{my}{e}(r-z) = (y-d)(sz-b^2)\frac{k}{hz} + A^2 \quad \ldots\ldots\ldots\ldots (1).$$

This equation expresses the relation between y and z when EG bisects ABCD. For simplicity, put

$$v^2 = (y-d)(sz-b^2)\frac{k}{hz} + A^2 = 2AEQ;$$

then $Eh = \frac{v^2}{y}$, and $\quad Ba : Aa = Eh : Ah = \frac{nv^2}{my}$;

therefore $\quad Qh = y - \frac{nv^2}{my} = \frac{my^2-nv^2}{my}$.

Again $\quad HL \cdot Fb = a^2$, or $Fb = \frac{a^2}{x}$;

and $\quad Ba : aL = Fb : bL = \frac{pa^2}{mx}$;

therefore $\quad Hb = x - \frac{pa^2}{mx} = \frac{mx^2-pa^2}{mx}$.

But the triangles EhQ and HFb are evidently similar, hence

$$Eh : hQ = Hb : Fb,$$

that is, $\quad \frac{v^2}{y} : \frac{my^2-nv^2}{my} = \frac{mx^2-pa^2}{mx} : \frac{a^2}{x}$,

or $\quad (my^2-nv^2)(mx^2-pa^2) = a^2v^2m^2 \quad \ldots\ldots\ldots\ldots (2).$

This is an equation for the relation between x and y, as v is always determinable in terms of y from equation (1), when EG, FH are perpendicular to each other, and bisect ABCD; that is, if the values of x, y, z are such as to satisfy equations (1) and (2), then AEPH will be equal to PFCG, and EBFP = HPGD.

There is yet, however, another condition to be satisfied, viz., that each of these parts shall be just one-fourth of the whole, and this will evidently be the case when any one of them is equal to that fourth part. Now to get an equation that will express this condition, draw Pk perpendicular to AD; then by similar triangles

$$Eh : Qh = Pk : Qk = Pk\left(\frac{my^2-nv^2}{mv^2}\right),$$

and $\quad Fb : Hb = Pk : Hk = Pk\left(\frac{mx^2-pa^2}{ma^2}\right)$;

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therefore $$HQ = Pk\left\{\frac{(my^2-nv^2)a^2+(mx^2-pa^2)v^2}{ma^2v^2}\right\}.$$

But $$HQ = AQ-AH = AQ-(c-x) = y+x-c;$$

therefore $$Pk = \frac{(y+x-c)ma^2v^2}{(my^2-nv^2)a^2+(mx^2-pa^2)v^2},$$

and $$\Delta HPQ = \frac{(y+x-c)^2ma^2v^2}{2(my^2-nv^2)a^2+2(mx^2-pa^2)v^2}.$$

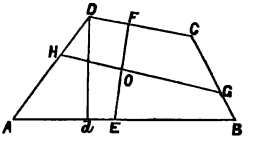
Therefore subtracting GDQ or $\frac{1}{2}(v^2-A^2)$, we have

$$HPGD = \frac{(y+x-c)^2ma^2v^2}{2(my^2-nv^2)a^2+2(mx^2-pa^2)v^2} - \frac{v^2-A^2}{2} = \frac{A^2}{4} \quad \ldots\ldots(3).$$

We have now three independent equations (1), (2), (3), involving the three unknowns x, y, z; any two of which may be eliminated. The resulting equation, however, will necessarily be a very complex one.

When the given figure is a triangle, the solution will be much simpler, and will terminate in an equation of the 8th degree.

II. *Solution by* SAMUEL BILLS.



Let ABCD be the given quadrilateral, and EF, GH the required straight lines, dividing it into four equal parts, and intersecting each other at right angles in O. Draw Dd perpendicular to AB; put $AB = l$, $Ad = m$, $Dd = n$, area of $ABCD = s$; and taking A as origin, let the rectangular equations of AB, BC, CD, DA, EF, GH be, respectively,

$$y = 0, \quad x = ay+b, \quad y = a_1x+b_1, \quad x = cy, \quad y = ux+v, \quad x = -uy+w;$$

so that EF is perpendicular to GH.

Then the coordinates of E, F, G, H, O are readily found to be

$$\left(-\frac{v}{u}, 0\right), \quad \left(\frac{v-b_1}{a_1-u}, \frac{a_1v-b_1u}{a_1-u}\right), \quad \left(\frac{aw+bu}{a+u}, \frac{w-b}{a+u}\right),$$

$$\left(\frac{cw}{c+u}, \frac{w}{c+u}\right), \quad \left(\frac{w-uv}{u^2+1}, \frac{v+uw}{u^2+1}\right);$$

and finding, from the coordinates, the areas of the quadrilaterals AEFD, ABGH, AEOH, we obtain the following equations:—

$$\frac{n(v-b_1)}{a_1-u} - \left(\frac{a_1v-b_1u}{a_1-u}\right)\left(\frac{v}{u}+m\right) = s \quad \ldots\ldots(1),$$

$$\left(\frac{aw+bu}{a+u}\right)\left(\frac{w}{c+u}\right) + \left(\frac{w-b}{a+u}\right)\left(l-\frac{cw}{c+u}\right) = s \quad \ldots\ldots(2),$$

$$\left(\frac{w-uv}{u^2+1}\right)\left(\frac{w}{c+u}\right) - \left(\frac{v+uw}{u^2+1}\right)\left(\frac{v}{u}+\frac{cw}{c+u}\right) = \tfrac{1}{2}s \quad \ldots\ldots(3).$$

From (1), (2), (3) we can find the values of u, v, w, which will determine the required positions of the dividing lines EF, GH.

If we assume $v = pu$, $w = q(c+u)$, and put

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$$L_1 = a_1u, \quad M_1 = a_1mu - b_1u - nu, \quad N_1 = a_1s + b_1n - su - b_1mu,$$
$$L_2 = cu - au - ac + c^2, \quad M_2 = -bc - bu - cl - lu, \quad N_2 = as + su + bl,$$
$$L_3 = 2u, \quad M_3 = 4u^2 + 4cu, \quad N_3 = 2c^2u + 2cu^2 - 2c - 2u, \quad Q_3 = -s - su^2;$$

then since $b = l$, we shall have

$$L_1 = a_1u, \quad M_1 = (a_1m - b_1 - n)\,u, \quad N_1 = u_1s + b_1n - (s + b_1m)\,u,$$
$$L_2 = (c - a)\,(c - u), \quad M_2 = 2l\,(c + u), \quad N_2 = us + l^2 + su,$$
$$L_3 = 2u, \quad M_3 = 4u\,(c + u), \quad N_3 = 2\,(cu - 1)\,(c + u), \quad Q_3 = -s\,(u^2 + 1);$$

and equations (1), (2), (3) will become

$$L_1p^2 + M_1p + N_1 = 0, \quad L_2q^2 + M_2q + N_2 = 0 \ldots\ldots\ldots\ldots (4, 5),$$
$$L_3p^2 + M_3pq + N_3q^2 = Q_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In (4), (5), (6) the unknowns are p, q, u, since L, M, N, Q are functions of u only; and substituting in (6) the values of p, q derived from (4), (5), we shall obtain an equation in u, from which u, and thence p, q, v, w, can be found; and thus the problem will be completely solved.

[A particular case of this problem will be found solved as Quest. 3288, on p. 105 of Vol. XV. of the *Reprint*.]

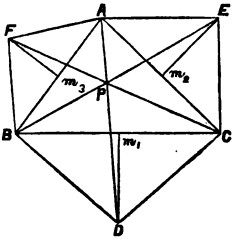
1281. (Proposed by the EDITOR.)—From the middle points of the sides of a triangle draw perpendiculars, proportional in length to these sides, and join the ends of the perpendiculars with the opposite corners of the triangle; then prove (1) that the joining lines will meet in a point, and (2) that, for varying lengths of perpendiculars, the locus of this point is the circumscribed rectangular hyperbola, whose trilinear equation is

$$\frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = 0.$$

Solution by STEPHEN WATSON.

Let m_1D, m_2E, m_3F be the perpendiculars, their trilinear equations in terms of the sides being

$$(b^2 - c^2)\,\alpha + ab\beta - ca\gamma = 0 \ldots\ldots (1),$$
$$(c^2 - a^2)\,\beta + bc\gamma - ab\alpha = 0 \ldots\ldots (2),$$
$$(a^2 - b^2)\,\gamma + ca\alpha - bc\beta = 0 \ldots\ldots (3).$$



Now $\angle ABF = \angle CBD$; hence the equations of BF, BD may be written

$$m\gamma + \alpha = 0, \quad \gamma + m\alpha = 0 \ldots\ldots (4, 5).$$

Eliminate γ from (1) and (5), the result is

$$(b^2 - c^2 + acm)\,\alpha + ab\beta = 0,$$

which is the equation of CD; and since $\angle BCD = \angle ACE$, the equation of CE must be

$$ab\alpha + (b^2 - c^2 + acm)\,\beta = 0 \ldots\ldots\ldots\ldots (6).$$

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Eliminate α, β, γ from the pairs of equations (1), (5); (2), (6); (3), (4), respectively, the results

$$\left.\begin{aligned} abm\beta - (b^2 - c^2 + acm)\gamma &= 0 \\ c(b^2 - c^2 + acm)\gamma + a(a^2 - b^2 - acm)\alpha &= 0 \\ (a^2 - b^2 - acm)\alpha + bcm\beta &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7)$$

are the equations of AD, BE, CF, which pass through the same point P, and the locus of P is found, by eliminating m from any two of the equations (7), to be $ab(a^2 - b^2)\alpha\beta + bc(b^2 - c^2)\beta\gamma + ca(c^2 - a^2)\gamma\alpha = 0$,

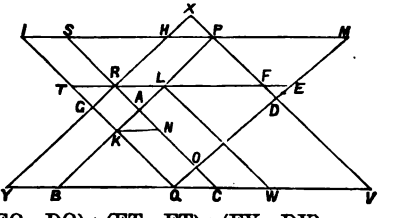
which represents a rectangular hyperbola passing through A, B, C.

[Putting sin A, sin B, sin C for a, b, c, Mr. WATSON'S equation of the locus takes the form given in the question, thus agreeing with the result obtained by the PROPOSER and Mr. WHITWORTH on pp. 111, 112 of Vol. XVIII. of the *Reprint*.]

2211. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.) — Prove that straight lines drawn through three points, parallel to three given straight lines, form three triangles, of which the perimeter of the greatest is equal to the sum of the perimeters of the other two.

Solution by Mr. W. HOPPS; A. B. EVANS, M.A.; *and others.*

Let ABC, DEF, GHI be three triangles formed by drawing three lines through each of the points P, Q, R parallel to three given lines; also, let S, T, V, K, L be the intersections of AC, PH; FE, QI; BC, DP; QI, AB; AB, FE; and draw KN, LW parallel to BC, CA.



Then, perimeter of $\triangle DEF = (EQ - DQ) + (ET - FT) + (FV - DV)$
$= (LB - KP) + (BW - QV) + (RC - NS)$.

Also, perimeter of $\triangle GHI = GR + RH + HS + SI + IT + TG$
$= AK + PL + WV + QC + SR + NA$.

Consequently, adding, we get

perimeter of $\triangle DEF$ + perimeter of $\triangle GHI$ =
$AK + (LB + PL) - KP + (BW + WV) - QV + QC + (RC + SR) - NS + NA$
$= AB + BC + CA$ = perimeter of $\triangle ABC$.

There are many cases of this theorem, but the proof is essentially the same in all. It will be found that the greatest triangle is always inverted with respect to the other two.

The theorem may be otherwise enunciated as follows:—

Three indefinite straight lines intersect and form a triangle, and *any* point is taken in each; through each point lines are drawn parallel to the indefinite lines in which such point is not situated, forming thereby other two triangles; then the perimeter of the greatest of these three triangles is equal to the sum of the perimeters of the other two.

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[The three triads of parallel lines will, by their intersections, form *two* distinct triads of similar triangles, one line being taken from each triad of parallels to form each triangle, and the *same* line used only once in each of the two triads of triangles. We have produced the lines of the diagram in order to show *both* these triads. If we call the three triads of parallels P_1, Q_1, R_1, (IM, YV, TE); P_2, Q_2, R_2, (BP, QM, YX); P_3, Q_3, R_3, (VX, QI, CS); the two triads of triangles formed by them will be

$$P_1Q_3R_2, \text{(GHI)}; \quad P_2Q_1R_3, \text{(ABC)}; \quad P_3Q_2R_1, \text{(DEF)} \ \ldots\ldots\ldots (1).$$

$$P_1Q_2R_3, \text{(SMO)}; \quad P_2Q_3R_1, \text{(KTL)}; \quad P_3Q_1R_2, \text{(XYV)} \ \ldots\ldots\ldots (2).$$

The property in the Question may be very simply proved of the *second* triad (2). For $YB=RL$, $BQ=PM$, $QC=TR$, $CV=SP$;

hence we have

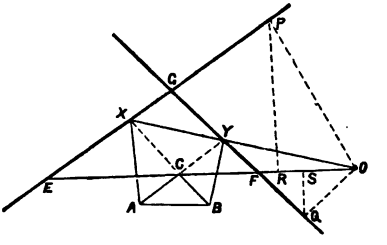
$$SM+TL = VY; \text{ also, } MO+LK = MQ+AL = HY+HX = YX,$$

and $$OS+KT = SC+KG = PV+PX = XV;$$

therefore perimeter of $\triangle XYV$ = per. of $\triangle SMO$ + per. of $\triangle KTL$.]

4063. (Proposed by Professor TOWNSEND, F.R.S.)—One side of a quadrilateral is fixed, the opposite side passes through a fixed point, and the two variable vertices move on fixed lines; construct it so as to have a given or maximum area.

Solution by the PROPOSER.



Let AXYB be the required quadrilateral, A and B its two fixed vertices, X and Y its two variable vertices, EG and FG the two fixed lines on which X and Y move, and O the fixed point through which XY passes; then, drawing AC and BC parallels through A and B to EG and FG, thus determining by their intersection the fixed point C, and joining CX and CY, the areas of the two variable triangles ACX and BCY being independent of the positions of X and Y on EG and FG, if the quadrilateral AXYB have a given or maximum area, so will also the triangle XCY; and the problem to construct the quadrilateral so as to have either, is consequently reduced to the corresponding one for the triangle, which is old and well known. The position of XY for the triangle of given area being determinable as a simple case of homographic division, and for the triangle of maximum area being such that $GY^2 : GX^2 = OR : OS$; where OP and QQ are perpendiculars from O on EG and FG, and PR and QS perpendiculars from P and Q on OC.

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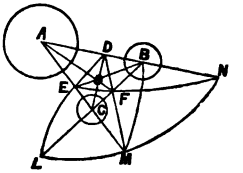
4097. (Proposed by Professor CLIFFORD.)—If about a prolate conicoid of revolution there be described an octahedron so that its three diagonals pass through a focus; show that they must be at right angles to each other.

Solution by Professor WOLSTENHOLME, M.A.

Reciprocating on the focus, this proposition only asserts that any parallelepiped inscribed in a sphere is rectangular.

2014. (Proposed by J. M'DOWELL, B.A., F.R.A.S.)—Prove that the centres of similitude of three small circles on the surface of a sphere lie three by three on four great circle-arcs (the axes of similitude).

Solution by the PROPOSER.



Let A, B, C be the centres of the three small circles; r, r', r'', their spherical radii; D, E, F their internal centres of similitude; L, M, N the external centres of similitude. Draw the great-circle arcs AF, BE, CD, LM, and MN, and join B, M by a great-circle arc. In fact, all the lines in the figure are great-circle arcs except the three small circles.

By known properties of the centres of similitude, we have

$$\sin AE : \sin EC = \sin r : \sin r''$$
$$= \left\{ \begin{matrix} \sin r & : \sin r' \\ \sin r' & : \sin r'' \end{matrix} \right\} = \left\{ \begin{matrix} \sin AD & : \sin DB \\ \sin BF & : \sin FC \end{matrix} \right\}.$$

Hence, by the first principles of spherical transversals, the three arcs AF, BE, CD pass through the same point O.

Now, if DF be produced, it will meet AEC in a point which will form with the three points A, E, and C, an harmonic range; but M also forms with these three points such a range; therefore, DF must pass through M. Similarly, DE must pass through L, and EF through N.

Again, since BL and AN are divided harmonically, M (BFCL) and M (ADBN) are harmonic pencils; hence LM and MN must form one continued great-circle arc.

The same method, of course, is applicable to the corresponding question on a plane.

[This theorem forms Prop. 33 of the Article on Spherical Transversals in the 2nd vol. of Davies's edition of Hutton's Course. Mr. Davies's proof is as follows:—

$$\sin AD : \sin DB = \sin r : \sin r', \quad \sin BF : \sin FC = \sin r' : \sin r'',$$
$$\sin CM : \sin MA = \sin r'' : \sin r;$$

therefore $\sin AD \cdot \sin BF \cdot \sin CM = \sin AM \cdot \sin BD \cdot \sin CF$;

therefore D, F, M are in one great circle. In like manner, it may be proved that DEL, EFN, LMN are continuous great-circle arcs.]

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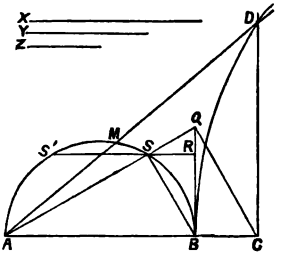
2215. (Proposed by the EDITOR.)—Produce a given straight line, so that the square on the whole line thus produced may be to the rectangle contained by the given line and the part produced in a ratio which is given, or a minimum.

Solution by the PROPOSER.



Let AB be the given line, which it is required to produce to a point C, so that

$$AC^2 : AB \cdot BC = X : Z.$$

Find Y a mean proportional between X and Z; draw BQ perpendicular to AB, and on BQ take BR a fourth proportional to X, Y, and AB; then

$$AB^2 : BR^2 = X^2 : Y^2 = X : Z.$$

On AB construct a semicircle, and draw RSS′ parallel to AB, cutting the semicircle in S, S′; join AS (or AS′) and produce it to meet BR in Q; and draw QC perpendicular to AQ; then will AB be produced to C as required.

For $$AC : AB = AQ : AS = BQ : BR;$$

therefore $AC^2 : BQ^2 = AB^2 : BR^2$, or $AC^2 : AB \cdot BC = X : Z$.

If $BR < \frac{1}{2}AB$, there are *two* points C, C′ (corresponding to S, S′) each of which will satisfy the conditions of the problem. If $BR > \frac{1}{2}AB$, the problem is *impossible*; and if $BR = \frac{1}{2}AB$, then RS *touches* the semicircle in its middle point M, AB = BC, and the ratio of AC^2 to AB . BC is a minimum, having 4 as its least value.

If we draw CD at right angles to AC and equal to AC, the point D will always be on the straight line AM; moreover, since $CD^2 : AB \cdot BC = X : Z$, when the ratio X : Z is given, D will also lie on a parabola whose vertex is B and parameter a fourth proportional to Z, X, AB. The value of X : Z will be a minimum when the given line AM just touches the parabola; and then the sub-tangent AC is equal to 2BC, and X = 4Z.

3988. (Proposed by C. TAYLOR, M.A.)—Two conics being inscribed in a quadrilateral, a variable conic has double contact with both of them; find the locus of intersection of the variable common tangents.

Solution by Professor WOLSTENHOLME, M.A.

In my *Book of Mathematical Problems* is the result that, "if a conic have double contact with each of two confocal conics, the common tangents are at right angles," and therefore the locus of their intersection is a circle. Of course, projecting the system, we have two conics inscribed in a quadrilateral, and a conic having double contact with both; and the circular locus becomes a conic passing through the ends of a diagonal of the quadrilateral, the pole of that diagonal being the intersection of the other two.

Any two conics being inscribed in some quadrilateral, we might alter the enunciation so as to make the proposition apparently (not really) more general.

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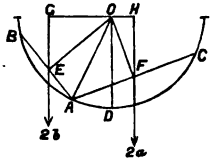
4048. (Proposed by ASHER B. EVANS, M.A.)—Two uniform beams whose lengths are $2a$ and $2b$ rest wholly within a hemispherical bowl with their ends against each other. If r be the radius of the bowl, ϕ the angle made with the vertical by the radius drawn to the junction of the beams, and the lineal unit of each beam be taken as the unit of weight, prove that the condition of equilibrium is $\tan\phi = \dfrac{a^2(r^2-a^2)^{\frac{1}{2}} - b^2(r^2-b^2)^{\frac{1}{2}}}{r^2(a+b) - a^3 - b^3}$.

Solution by A. MARTIN; W. SIVERLEY; *the* PROPOSER; *and others.*

Put $AC = 2a$, $AB = 2b$, $\angle AOF = \alpha$, $\angle AOE = \beta$;



then $OF = (r^2-a^2)^{\frac{1}{2}}$, $OE = (r^2-b^2)^{\frac{1}{2}}$,

$OH = (r^2-a^2)^{\frac{1}{2}} \sin(\alpha-\phi)$,

$OG = (r^2-b^2)^{\frac{1}{2}} \sin(\beta+\phi)$.

Taking moments about O, we have

$$2a \,.\, OH = 2b \,.\, OG;$$

therefore $$a(r^2-a^2)^{\frac{1}{2}} \sin(\alpha-\phi) = b(r^2-b^2)^{\frac{1}{2}} \sin(\beta+\phi);$$

whence $$\tan\phi = \frac{a(r^2-a^2)^{\frac{1}{2}} \sin\alpha - b(r^2-b^2)^{\frac{1}{2}} \sin\beta}{a(r^2-a^2)^{\frac{1}{2}} \cos\alpha + b(r^2-b^2)^{\frac{1}{2}} \cos\beta}.$$

But $\sin\alpha = \dfrac{a}{r}$, $\cos\alpha = \dfrac{(r^2-a^2)^{\frac{1}{2}}}{r}$, $\sin\beta = \dfrac{b}{r}$, $\cos\beta = \dfrac{(r^2-b^2)^{\frac{1}{2}}}{r}$.

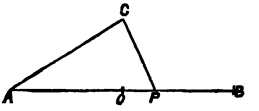
Making these substitutions, we get

$$\tan\phi = \frac{a^2(r^2-a^2)^{\frac{1}{2}} - b^2(r^2-a^2)^{\frac{1}{2}}}{r^2(a+b) - a^3 - b^3}.$$

1160. (Proposed by the EDITOR.)—The sum of two sides of a triangle is constant and equal to $2a$; show that if all possible values of these sides, and of their included angle, be supposed equally probable, the mean value of the third side is $\frac{1}{2}\pi a$.

Solution by STEPHEN WATSON.

Let $AO = OB = a$ be half the given sum of the sides; P any point in OB; $PC = PB$; $OP = x$, and $\angle CPA = 2\theta$. Then $AP = a + x$, $BP = CP = a - x$, and



$$AC = (AP^2 + CP^2 - 2AP \,.\, CP\cos 2\theta)^{\frac{1}{2}}$$
$$= 2\cos\theta(a^2\tan^2\theta + x^2)^{\frac{1}{2}}$$

Now an element at C is $(a-x)\,2d\theta$, and an element at P is dx; hence the whole number of triangles is expressed by

$$2\int_0^{\frac{1}{2}\pi}\int_a^a (a-x)\,d\theta\,dx = 2\pi a^2,$$

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the point C being considered as always lying above AB, and P as taking any position between O and B; and these are all the positions we need take into account in order to obtain the true average. Hence the

average is $$\frac{2}{\pi a^2}\int_0^{\frac{1}{2}\pi}\int_{-a}^{a}(a-x)\,(a^2\tan^2\theta+x^2)^{\frac{1}{2}}\cos\theta\,d\theta\,dx.$$

[At this point the result coincides with that obtained by the EDITOR, in his solution of this Question, on p. 19 of Vol. III. of the *Reprint*, eq. (2), where the above definite integral is worked out at full length, and shown to lead to the value of the average stated in the Question.]

4001. (Proposed by FRANCIS GALTON.)—A large nation, of whom we will only concern ourselves with the adult males, N in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, a_0 per cent. of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and so on, up to a_5 who have five. Find (1) what proportion of the surnames will have become extinct after r generations; and (2) how many instances there will be of the same surname being held by m persons.

I. *Solution by the* Rev. H. W. WATSON, M.A.

Let the enunciation of the problem be generalised by substituting some general symbol, as q, for 5.

Let $$\frac{a_0}{100}=t_0,\quad \frac{a_1}{100}=t_1,\ \&c.,\quad \frac{a_q}{100}=t_q;$$

i.e., let $t_0, t_1,$ &c. ... t_q represent the respective chances of any individual man having no, one, &c. up to q, sons who reach adult life, these chances being the same in every generation.

If any given surname be held by p males in any one generation, it follows, from the ordinary formulæ in the doctrine of chances, that the chance of this surname being held by s males exactly in the next succeeding generation will be the coefficient of x^s in $(t_0+t_1x+\ldots+t_qx^q)^p$.

Let $t_0+t_1x\ldots+t_qx^q$ be denoted by the symbol T.

Also, let the number of surnames held by s men exactly in the rth generation be denoted by the symbol ${}_rm_s$.

Then, from what has been last stated, we have ${}_rm_s$ = coefficient of x_s in the expression

$${}_{r-1}m_0+{}_{r-1}m_1\mathrm{T}+{}_{r-1}m_2\mathrm{T}^2\ldots\ldots+{}_{r-1}m_{q^{r-1}}\mathrm{T}^{q^{r-1}},$$

omitting the factor expressing the number of men at starting.

Let $${}_{r-1}m_0+{}_{r-1}m_1\cdot\mathrm{T}+{}_{r-1}m_2\cdot\mathrm{T}^2+\&c.+{}_{r-1}m_{q^{r-1}}\cdot\mathrm{T}^{q^{r-1}}$$

be denoted by $f_r(x)$, then it follows, from the same reasoning, that ${}_{r-1}m_1$, ${}_{r-1}m_2$ &c., ${}_{r-1}m_p$ &c., are the coefficients of x, x^2 &c. ... x^p in the expression $f_{r-1}(x)$.

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Therefore, if a series of functions of x be formed, such that

$$f_1(x) = t_0 + t_1 x \ldots + t_q x^q, \quad \text{and} \quad f_r(x) = f_{r-1}(t_0 + t_1 x \ldots + t_q x^q),$$

then the number of groups of surnames with s representatives in the rth generation is the coefficient of x^s in $f_r(x)$, and the number of surnames extinguished in that time will be the term independent of x in $f_r(x)$.

As an easy example, suppose that $q = 2$, and that we require the number of names extinguished after three generations.

We have to find the term independent of x in $f_3(x)$.

Here
$$f_1(x) = (t_0 + t_1 x + t_2 x^2),$$
$$f_2(x) = t_0 + t_1(t_0 + t_1 x + t_2 x^2) + t_2(t_0 + t_1 x + t_2 x^2)^2,$$
$$= (t_0 + t_1 t_0 + t_2 t_0^2) + (t_1^2 + 2t_0 t_1)x + (t_1 t_2 + t_2 t_1^2 + 2t_0 t_2^2)x^2 + 2t_2^2 t_1 x^3 + t_2^3 x^4,$$
$$f_3(x) = (t_0 + t_1 t_0 + t_2 t_0^2) + (t_1^2 + 2t_0 t_1)(t_0 + t_1 x + t_2 x^2)$$
$$+ (t_1 t_2 + t_2 t_1^2 + 2t_0 t_2^2)(t_0 + t_1 x + t_2 x^2)^2 + 2t_2^2 t_1 (t_0 + t_1 x + t_2 x^2)^3$$
$$+ t_2^3 (t_0 + t_1 x + t_2 x^2)^4.$$

The number of names extinct is therefore

$$t_0 + t_1 t_0 + t_2 t_0^2 + t_0(t_1^2 + 2t_0 t_1) + (t_1 t_2 + t_2 t_1^2 + 2t_0 t_2^2) t_0^2 + 2t_0^3 t_2^2 t_1 + t_2^3 t_0^4.$$

II. *Solution by* G. S. Carr.

1. Let $\frac{a_0}{100} = t$, $\frac{a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5}{100} = p$, $\frac{a_1 + a_2 + a_3 + a_4 + a_5}{100} = q$,

so that $t + q = 1$; then the variation in the population will be as exhibited in the following table:—

No. of the generation.	No. of persons living at its close.	No. of different surnames remaining.	No. of surnames that become extinct in each generation.
1	pN	qN	tN
2	p^2N	q^2N	tqN
3	p^3N	q^3N	tq^2N
...	...	...	...
r	p^rN	q^rN	tq^{r-1}N

Thus the number of surnames that will have become extinct after r generations have died out will be $\mathrm{N}(1 - q^r)$.

2. The number of persons living at the end of the rth generation $= p^r$N. The product of the coefficients 1, 2, 3, 4, 5 and their powers which occurs in any term of the expansion of $(a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5)^r$ expresses a certain number of persons having the same name at the close of the rth generation; and the remaining factors of that term multiplied by $\frac{\mathrm{N}}{100^r}$ signifies the number of existing groups of such persons.

Hence the number of instances in which m or more persons will hold the same name after r generations have elapsed will be found by taking the sum of certain terms in the expansion of

$$(a_1 + a_2 + a_3 + a_4 + a_5)^r \mathrm{N} \div 100^r;$$

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such terms being of the form

$$\frac{\lfloor r}{\lfloor\alpha \lfloor\beta \lfloor\gamma \lfloor\delta \lfloor\epsilon} a_1^{\alpha} a_2^{\beta} a_3^{\gamma} a_4^{\delta} a_5^{\epsilon} \cdot \frac{N}{100^r},$$

and subject to the two conditions $\alpha+\beta+\gamma+\delta+\epsilon=r$ and $1^{\alpha}.2^{\beta}.3^{\gamma}.4^{\delta}.5^{\epsilon} > m$, α, β, γ, δ, ϵ being positive integers.

If the same surname is held by m persons *exactly*, m must be of the form $2^{\beta+2\delta}.3^{\gamma}.5^{\epsilon}$. γ and ϵ have now fixed values, and the only variations in the values of α, β, and δ are those allowed by the two equations

$\alpha+\beta+\delta = r-\gamma-\epsilon$ a given integer, and $\beta+2\delta =$ a given integer.

Mr. GALTON rightly remarks that the foregoing Solution is not satisfactory.

If $\frac{a_0}{100} = t$, then $t =$ the probability of Jones having no issue, $t^2 =$ the probability of two Joneses both having no issue, $t^3 =$ the probability of three Joneses having no issue, and so on.

Hence, if M surnames be each held by three persons, the number of names that will become extinct out of them in one generation is t^3M. This method leads to a different result, and is undoubtedly correct in principle. But to obtain any concise algebraic statement of the process in this case seems, from the nature of the problem, to be very difficult, if not impossible.

3909. (Proposed by Professor TOWNSEND, F.R.S.)—The interior of a homogeneous shell, bounded internally and externally by similar and similarly placed but not concentric ellipsoids, and attracting according to the law of the inverse square of distance, is partially filled with homogeneous fluid; given all particulars, determine the form and position of the free surface of the latter under the action of the former.

Solution by the PROPOSER.

Denoting, by α, β, γ the coordinates of the centre of the inner with respect to the principal planes of the outer ellipsoid, by A, B, C the coefficients of the three components of the interior attraction of either solid for the given law of force, and by X, Y, Z the components of the attraction of the shell for any point xyz interior to both its surfaces; then, since for any point xyz interior to both surfaces $X=A\alpha$, $Y=B\beta$, $Z=C\gamma$, therefore for every point xyz of the free surface of the fluid

$$A\alpha \,.\, dx + B\beta \,.\, dy + C\gamma \,.\, dz = 0,$$

and consequently $A\alpha \,.\, x + B\beta \,.\, y + C\gamma \,.\, z =$ a constant; thus showing that the free surface is a plane, whose direction angles depend on and are given with the six constants A, B, C and α, β, γ, and whose distance from the origin depends on and is given with the volume of the fluid.

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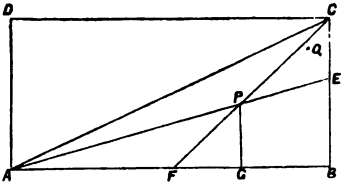
3467. (Proposed by S. WATSON.)—Two points are taken at random within a given rectangle; show that the chance of the straight line passing through them intersecting opposite sides is $\frac{2}{3}$.

Solution by the PROPOSER.

Let ABCD be the rectangle; P any point in the triangle ABC, through which draw AE, CF to meet BC, BA in E, F, and draw PG perpendicular to AB. Put $AB=a$, $BC=b$, $AG=x$, $GP=y$. In order that PQ may intersect the adjacent sides AB, BC, Q must lie in one of the triangles APF, CPE. Now we have



$$\triangle APF = \tfrac{1}{2}AF \cdot y = \frac{(bx-ay)\,y}{2(b-y)}, \quad \triangle CPE = \tfrac{1}{2}(a-x)\left(\frac{bx-ay}{x}\right) \ldots\ldots (1, 2).$$

Multiplying by 4, because any pair of adjacent sides may be intersected, and dividing by a^2b^2, the number of ways the two points can be taken within the triangle, the chance of intersecting adjacent sides

$$\frac{4}{a^2b^2}\int_0^a dx \int_0^{\frac{bx}{a}} dy \left\{(1)+(2)\right\} = \tfrac{1}{3};$$

therefore the chance of intersecting opposite sides is $\frac{2}{3}$.

3997. (Proposed by W. C. OTTER, F.R.A.S.)—Suppose a man has a calf which at the end of three years begins to breed, and afterwards brings forth a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow calf every year; and that these last breed in the same manner, &c. Required the owner's stock at the end of x years.

Solution by W. SIVERLY; Rev. J. L. KITCHIN, M.A.; *and others.*

Let u_x = the number at the end of x years; then we have the series 1, 1, 2, 3, 4, 6, 9, 13, 19, &c.;

hence
$$u_{x+3} - u_{x+2} - u_x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now let r_1, r_2, r_3 be the three roots of the cubic equation $z^3 - z^2 - 1 = 0$; then the complete integral of (1) is

$$u_x = c_1 (r_1)^x + c_2 (r_2)^x + c_3 (r_3)^x \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where c_1, c_2, c_3 are constants, which may be found from the three equations

$$u_0 = c_1 + c_2 + c_3 = 1, \quad u_1 = c_1 r_1 + c_2 r_2 + c_3 r_3 = 1, \quad u_2 = c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2 = 1.$$

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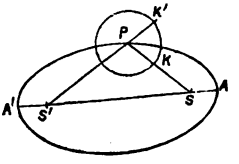
3992. (Proposed by A. M. NASH.)—S is a focus of a conic, P any point on the curve; with centre P, and radius equal to $SP-k$, k being any constant, a circle is drawn; show that its envelope consists of a pair of circles whose centres are at the foci, and whose radii are k and $2a \mp k$, where $2a$ is the major axis, the upper sign applying to the ellipse and the lower to the hyperbola; and that in the parabola the latter circle becomes a straight line parallel to the directrix and at a distance k from it.

Solution by the PROPOSER.



Let S′ be the other focus; join SP, S′P, and let K, K′ be the intersections of these lines with the circle, one of the points K, K′ being inside, the other outside the conic.

With centres S, S′, and radii SK, S′K′, draw circles; then the circle whose centre is P will touch both of these circles. Now we have

$$SK = SP - PK = k,$$

$$S'K' = S'P \pm PK' = S'P \pm (SP - k) = AA' \mp k,$$

the upper or lower sign being taken, according as the curve is an ellipse or hyperbola. Hence these circles are fixed, and therefore they are the envelope of the system of circles.

If the conic be a parabola, draw PM perpendicular to the directrix, and let SP, PM cut the circle in K, K′. Then

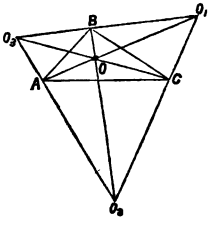
$$MK' = PM - PK' = SP - PK = k,$$

and SP, PM cut the circle at right angles; hence the envelope is a circle whose centre is S and radius k, and a line parallel to the directrix and at a distance k from it.

4082. (Proposed by T. T. WILKINSON, F.R.A.S.)—In any triangle ABC, if O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles respectively, prove that $AO \cdot AO_1 \cdot AO_2 \cdot AO_3 = AB^2 \cdot AC^2$,

$BO \cdot BO_1 \cdot BO_2 \cdot BO_3 = BA^2 \cdot BC^2$, $CO \cdot CO_1 \cdot CO_2 \cdot CO_3 = CB^2 \cdot CA^2$.

Solution by A. B. EVANS, M.A.; J. MERRICK; A. RENSHAW; *and others.*



From the similar triangles AOB and ACO_1, we have $AO \cdot AO_1 = AB \cdot AC$;

and from the similar triangles AOO_2 and AO_3O_1 we have $AO \cdot AO_1 = AO_2 \cdot AO_3$;

$\therefore$ $AO \cdot AO_1 = AO_2 \cdot AO_3 = AB \cdot AC$ ..... (1).

Similarly $BO \cdot BO_1 = BO_2 \cdot BO_3 = BA \cdot BC$...(2),

and $CO \cdot CO_1 = CO_2 \cdot CO_3 = CB \cdot CA$..... (3).

The required conditions follow immediately from (1), (2), (3).

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3936. (Proposed by S. TEBAY, B.A.)—Along the arc of a circle n equal particles are distributed at equal distances, the first being fixed. Find the locus of their centre of gravity, when the distance between the particles is supposed to vary.

Solution by the Rev. J. L. KITCHIN, M.A.; A. M. NASH; *and others.*

Let ϕ be the angle which two adjacent particles subtend at the centre; and let the axis of x be a diameter making an angle ϕ with a radius through the fixed particle, and the axis of y the perpendicular diameter; then, if $(\bar{x}, \bar{y})$ and (ρ, θ) be the coordinates of the centre of gravity, we have

$$n\bar{x} = x_1 + x_2 + x_3 + \ldots + x_n = r(\cos\phi + \cos 2\phi + \ldots + \cos n\phi)$$

or
$$n\rho\cos\theta = r\cos\tfrac{1}{2}(n+1)\phi\sin\tfrac{1}{2}n\phi\operatorname{cosec}\tfrac{1}{2}\phi,$$

$$n\bar{y} = y_1 + y_2 + \ldots + y_n = r(\sin\phi + \sin 2\phi + \ldots + \sin n\phi)$$

or
$$n\rho\sin\theta = r\sin\tfrac{1}{2}(n+1)\phi\sin\tfrac{1}{2}n\phi\operatorname{cosec}\tfrac{1}{2}\phi;$$

whence
$$\tan\theta = \tan\tfrac{1}{2}(n+1)\phi, \quad \text{or} \quad \phi = \frac{2\theta}{n+1},$$

and
$$n^2\rho^2 = r^2\sin^2\tfrac{1}{2}n\phi\operatorname{cosec}^2\tfrac{1}{2}\phi;$$

therefore the equation of the required locus is $n\rho\sin\dfrac{\theta}{n+1} = r\sin\dfrac{n\theta}{n+1}$.

2022. (Proposed by Professor CLIFFORD.)—A tangent to an ellipse is a chord of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters.

Solution by N'IMPORTE.

Let the equations to the ellipse, the circle, and the tangent at the point (h, k) be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 + y^2 = a^2 + b^2, \quad \frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \ldots\ldots (1, 2, 3).$$

Then the intersections of the tangent with the circle will be given by the simultaneous equations (2), (3); hence, eliminating y and x separately from (2), (3), and putting (x_1, x_2), (y_1, y_2) for the roots of the resulting equations, we have

$$\frac{x_1x_2}{x^4} = \frac{b^4 - (a^2+b^2)k^2}{a^4k^2 + b^4h^2}, \quad \frac{y_1y_2}{b^4} = \frac{a^4 - (a^2+b^2)h^2}{a^4k^2 + b^4h^2};$$

therefore
$$a^2y_1y_2 + b^2x_1x_2 = 0, \quad \text{or} \quad \frac{y_1y_2}{x_1} = -\frac{b^2}{a^2};$$

hence the lines joining the centre with the points (x_1, y_1), (x_2, y_2), (that is, with the ends of the chord,) are conjugate diameters.

NOTE.—If we change b^2 into $-b^2$, we obtain a similar theorem for the

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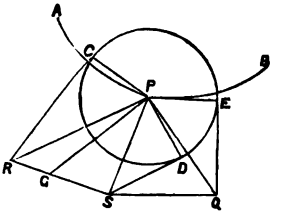
hyperbola (that is, the square on the radius of the circle is equal to the sum of the squares on the semi-axes in the ellipse, and to their difference in the hyperbola), but the conjugate diameters will be imaginary if

$$(a^4k^2 + b^4h^2)(a^2 - b^2) < a^4b^4.$$

2026. (Proposed by T. T. Wilkinson, F.R.A.S.)—In a line of any order, given in position, determine geometrically a point P, such that if a circle be drawn from P as a centre, with a given radius, and the tangents RC, SD, QE, &c., be drawn from any number of given points R, S, Q, &c., we may have $m \cdot RC^2 + n \cdot SD^2 + p \cdot QE^2 + \&c.$, equal to a given quantity.

I. *Solution by the late* Henry Buckley.



Analysis. Suppose the point P found in the line AB, as required; and also the circle drawn from P as centre with the given radius. From the points R, S, Q, &c., draw the tangents RC, SD, QE, &c.; draw also the radii CP, DP, PE, and join RP, SP, QP, &c. When there are two given points, we have

$$m \cdot CR^2 = m \cdot PR^2 - m \cdot CP^2,$$

and $$n \cdot SD^2 = n \cdot PS^2 - n \cdot PD^2;$$

whence $m \cdot PR + n \cdot PS^2 - m \cdot CP^2 - n \cdot PD^2$,

or $m \cdot PR^2 + n \cdot PS^2$ is equal to a given quantity. Again, join R, S; and divide it in G, so that $GS : RG = m : n$, and join GP; then (by Leslie's *Geometrical Analysis*, p. 323) $m \cdot RP^2 + n \cdot SP^2 = n \cdot RS \cdot SG + (m+n)\, PG^2$ is equal to a given quantity, or $(m+n)\, PG^2$ is equal to a given quantity. Hence the locus of P is a given circle, which will intersect the line AB in P, the point required.

When there are three given points, we have $m \cdot CR^2 = m \cdot RP^2 - m \cdot CP^2$, $n \cdot SD^2 = n \cdot SP^2 - n \cdot DP^2$, and $p \cdot QE^2 = p \cdot QP^2 - p \cdot PE^2$; whence $m \cdot RP^2 + n \cdot SP^2 + p \cdot QP^2$ is equal to a given quantity; but from above,

$$m \cdot RP^2 + n \cdot SP^2 = n \cdot RS \cdot SG + (m+n)\, PG^2;$$

therefore $n \cdot RS \cdot SG + (m+n)\, PG^2 + p \cdot QP^2$ is equal to a given quantity; or $(m+n)\, PG^2 + p \cdot QP^2$ is equal to a given quantity; hence (by Prop. 4, Lib. ii. of Apollonius *de Locis Planis*, restored by Dr. Simson) the locus of P is a given circle which will intersect the line AB in P, the point required. In the same manner the solution may be extended to any number of given points, as is manifest from what is done above.

Note.—If, instead of the sum of the squares of the tangents, their difference had been given; or, $m \cdot RC^2 - n \cdot SD^2 - p \cdot QE^2 - \&c.$ had been equal to a given space, the problem might have been constructed with nearly the same facility. Or if the ratio of the tangents RC, SD, QE, &c., $(= m : n, \&c.)$ had been given, the locus will still be a given circle. And lastly, in the case of a maximum or a minimum, the limits of possibility may be found by describing a circle from a given centre to touch the line AB given in position.

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II. *Solution by the* EDITOR.

The radius of the circle (P) being given, $m \cdot RP^2 + n \cdot SP^2 + p \cdot QP^2 + \&c.$ is given; and, if O be the centroid of the system of points R, S, Q, &c., ("centre de moyennes distances" of CARNOT), $m \cdot RO^2 + n \cdot SO^2 + p \cdot QO^2 + \&c.$ is also a given magnitude. But, by a known property of the centroid,

$$(m \cdot RP^2 + n \cdot SP^2 + p \cdot QP^2 + \&c.) = (m \cdot RO^2 + n \cdot SO^2 + p \cdot QO^2 + \&c.) + (m + n + p + \&c.)\, OP^2;$$

and hence OP is given.

The locus, then, of the point P, for which $m \cdot RP^2 + n \cdot SP^2 + p \cdot QP^2 + \&c.$ remains constant, is a sphere, whose centre is O and radius OP (or a circle, if the entire system is limited to one plane); and the intersections of this sphere (or circle) with the "line of any order given in position," determine the positions of the required point.

3993. (Proposed by M. COLLINS, B.A.)—If A, B, C be the vertices of a hypocycloid of three branches described by a point on the circumference of a circle which rolls within a fixed circle whose diameter is three times that of the rolling circle; and if A′BC be an equilateral triangle, A′ and A being upon opposite sides of BC, P being any point in the hypocycloid, PE perpendicular to BC, and PD a tangent to the circle whose centre is A′ and radius is A′B or A′C; prove that $PD^4 \div PE^3$ will be equal to sixteen times the diameter of the rolling circle.

I. *Solution by* C. LEUDESDORF.

Let $\angle O'OP = \phi$, $\angle O'OA = \theta$,

then $\angle OO'P = \pi - 3\theta$,

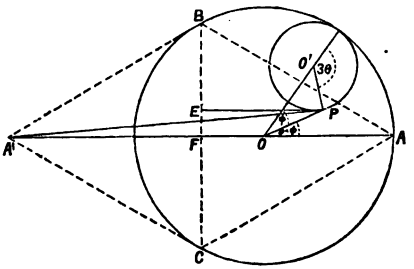
and

$$\frac{\sin\phi}{O'P} = \frac{\sin 3\theta}{OP} = \frac{\sin(3\theta - \phi)}{OO'};$$

or, taking the radius of the rolling circle as unity,

$$\frac{\sin\phi}{1} = \frac{\sin 3\theta}{OP} = \frac{\sin(3\theta - \phi)}{2};$$

$$\therefore \quad OP = \frac{\sin 3\theta}{\sin\phi} \quad \ldots\ldots (1),$$



and

$$\sin 3\theta \cot\phi = 2 + 3\cos\theta \quad \ldots\ldots (2).$$

Now $PD^2 = A'P^2 - A'C^2 = A'O^2 + OP^2 + 2A'O \cdot OP \cos(\theta - \phi) - AC^2$

$$= 36 + \frac{\sin^2 3\theta}{\sin^2\phi} + 12\frac{\sin 3\theta}{\sin\phi}\cos(\theta - \phi) - 27, \text{ by (1)},$$

$$= 9 + \sin^2 3\theta\,(1 + \cot^2\phi) + 12\sin 3\theta\,(\cos\theta\cot\phi + \sin\theta)$$

$$= 9 + \sin^2 3\theta + (2 + 3\cos\theta)^2 + 12\sin\theta\sin 3\theta + 12\cos\theta\,(2 + 3\cos\theta), \text{ by (2)},$$

$$= 2\,(1 + 2\cos\theta)^2.$$

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$$PE = OF + OP\cos(\theta - \phi)$$
$$= \tfrac{9}{2} - 3 + \frac{\sin 3\theta}{\sin\phi}\cos(\theta-\phi), \text{ by (1)},$$
$$= \tfrac{3}{2} + \sin 3\theta(\cos\theta\cot\phi + \sin\theta)$$
$$= \tfrac{3}{2} + \sin 3\theta\sin\theta + \cos\theta(2 + 3\cos\theta), \text{ by (2)},$$
$$= \tfrac{1}{2}(1 + 2\cos\theta)^2;$$

therefore $PD^4 = 4(1 + 2\cos\theta)^6 = 32PE^3$, which proves the theorem.

II. *Solution by* N'IMPORTE.

Taking the axis of x at right angles to BC, and the radius of the rolling circle as unity, the equations of the hypocycloid are

$$x = 2\cos\theta + \cos 2\theta, \quad y = 2\sin\theta - \sin 2\theta \ldots\ldots\ldots\ldots (1),$$

and that to the circle whose centre is A′ is

$$(x + 6)^2 + y^2 - 27 = 0 \ldots\ldots\ldots\ldots (2),$$

and by substituting in the left-hand member of (2) the values of x and y from (1), we have

$$PD^2 = x^2 + y^2 + 12x + 9 = 5 + 4\cos 3\theta + 24\cos\theta + 12\cos 2\theta + 9$$
$$= 16\cos^3\theta + 24\cos^2\theta + 12\cos\theta + 2 = 16(\cos\theta + \tfrac{1}{2})^3.$$

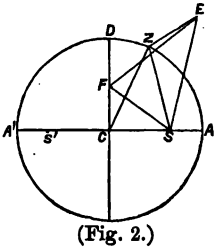
Again, $PE = x + \tfrac{1}{2} = 2\cos^2\theta + 2\cos\theta + \tfrac{1}{2} = 2(\cos\theta + \tfrac{1}{2})^2$;

therefore $PD^4 = 256(\cos\theta + \tfrac{1}{2})^6 = 32\,PE^3$, which proves the theorem.

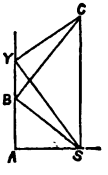
2012. (Proposed by J. MCDOWELL, B.A., F.R.A.S.)—A line is drawn from the focus of a conic to meet the tangent at a constant angle; find the locus of the point of intersection, and show that, in the case of the parabola, the locus will always touch it in one point, in that of the hyperbola in two points, but in the case of the ellipse it will only touch it in one or two points, under certain conditions.

Solution by the PROPOSER.

First, let S (Fig. 1) be the focus, A the vertex, and AY the tangent at the vertex of a parabola, YC any tangent to the curve, and therefore CYS a right-angle. Draw SC meeting the tangent at the ∠ YCS = the given angle. Also make ∠ABS = YCS; therefore the quadrilateral CYBS is circumscribable by a circle, and therefore the ∠SBC = SYC = a right angle; therefore BC is a fixed tangent to the parabola, since



(Fig. 2.)



(Fig. 1.)

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the point B and the line SB are fixed. This line BC is therefore the required locus.

Secondly (Fig. 2), let A′A be the major axis of an ellipse, C its centre, CD the direction of its minor axis, and A′DA the auxiliary circle, S_1, S the two foci, ZE any tangent to the ellipse, and therefore SZE = a right angle, E a point on the required locus. Draw SF making the angle CFS = SEZ, and join CZ, FE. The right-angled triangles FCS, EZS are obviously similar; therefore CS : SF = ZS : SE, and therefore CS : SZ = FS : SE, and ∠ CSZ = FSE, therefore the triangles CSZ and FSE are similar. Hence CS : CZ = FS : FE. But CS, CZ, and FS are all given in magnitude; therefore FE is of a known magnitude, and F is a fixed point.

Therefore, in the case of the ellipse, the required locus is a circle with its centre in the minor axis; and, in the same manner, by using the auxiliary circle of the hyperbola, the locus can be shown to be a circle with its centre on the transverse axis.

Since in every case the required locus must lie outside the curve, and two lines drawn from the foci of a central conic to any point on it make equal angles with the tangent at that point, it is plain that if on SS′, the straight line joining the foci of the ellipse, we draw a segment of a circle containing an angle equal to the supplement of twice the given angle, this segment will either cut the ellipse in two points, or touch it at the extremity of the minor axis, or not meet the ellipse at all; and that the point or points, if any, common to the ellipse and segment, are the points at which the locus touches the curve.

For the hyperbola, draw a segment of a circle on S′S containing twice the given angle (which we may always suppose acute). This segment will always cut the hyperbola in the two points where the required locus touches the curve.

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